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Definition of Symmetry of Determinants

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Abstract
Provides a definition of symmetry for determinants, which simplifies the proof of two results for determinants.

Keywords: Inverse Order Number, Determinant, Transposed Determinant, Cofactor, Algebraic Cofactor.

The Symmetry Definition of a Determinant

There are n^2 numbers arranged in a table with n rows n columns

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

The a_{ij} is called the element of D ,it has two subscripts. The first subscript i is called the row subscript, and the second subscript j is called the column subscript. The row subscript i and column subscript j indicate that the element is the row i column j element of this D, for example, a_{21} is the second row first column element of D.

Make the n product of the elements located in different rows and different columns in the table D, and label it with a symbol $(-1)^t$, obtain the form as follows

$$(-1)^t a_{p_1q_1}a_{p_2q_2}\cdots a_{p_nq_n} \qquad (1)$$

is called the items, which $p_1p_2\cdots p_n$ is an arrangement of natural numbers $1,2,\cdots,n$, $q_1q_2\cdots q_n$ is also an arrangement of natural numbers $1,2,\cdots,n$

$t=\tau(p_1p_2\cdots p_n)+\tau(q_1q_2\cdots q_n)$ where $\tau(p_1p_2\cdots p_n)$ is the inverse order number of the arrangement $p_1p_2\cdots p_n$, $\tau(q_1q_2\cdots q_n)$ is the inverse order number of the arrangement $q_1q_2\cdots q_n$.

Because arrangements $p_1p_2\cdots p_n$ have $n!$ items, $q_1q_2\cdots q_n$ also have $n!$ items , so items in the form of (1) have $(n!)^2$ items .Algebraic sum of the $(n!)^2$ items and division by $n!$, i.e

$\frac{1}{n!}\sum(-1)^t a_{p_1q_1}a_{p_2q_2}\cdots a_{p_nq_n}=D$ is called an n order determinant, denoted as

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_n \\ q_1q_2\cdots q_n}}(-1)^{\tau(p_1p_2\cdots p_n)+\tau(q_1q_2\cdots q_n)}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_nq_n}$$

Where, a_{ij} as mentioned earlier, are the (i,j) elements of this determinant D, that is, the elements of the row i and column j of this determinant D.

Assuming $i_1\cdots i_k\cdots i_l\cdots i_n$ is a fixed arrangement, exchange i_k,i_l obtain a new fixed arrangement $i_1\cdots i_l\cdots i_k\cdots i_n$, then

$$\sum_{q_1\cdots q_k\cdots q_l\cdots q_n}(-1)^{\tau(i_1\cdots i_k\cdots i_l\cdots i_n)+\tau(q_1q_2\cdots q_n)}a_{i_1q_1}\cdots a_{i_kq_k}\cdots a_{i_lq_l}\cdots a_{i_nq_n}$$

$$=\sum_{q_1\cdots q_l\cdots q_k\cdots q_n}(-1)^{\tau(i_1\cdots i_l\cdots i_k\cdots i_n)+\tau(q_1q_2\cdots q_n)}a_{i_1q_1}\cdots a_{i_lq_l}\cdots a_{i_kq_k}\cdots a_{i_nq_n}$$

This is because each permutation changes the parity of the arrangement and the multiplication of numbers satisfies the commutative law. Pay attention to that in each of the $n!$ arrangements here, q_k, q_l has been transformed into q_l, q_k, the arrangement $q_1\cdots q_k\cdots q_l\cdots q_n$ has been transformed into $q_1\cdots q_l\cdots q_k\cdots q_n$. where the q_k, q_l positions remain unchanged, and their values are swapped and vary with different arrangements! It can be inferred from above that

$$\sum_{q_1q_2\cdots q_n}(-1)^{\tau(i_1\cdots i_k\cdots i_l\cdots i_n)+\tau(q_1q_2\cdots q_n)}a_{i_1q_1}\cdots a_{i_kq_k}\cdots a_{i_lq_l}\cdots a_{i_nq_n}$$

$$=\sum_{q_1q_2\cdots q_n}(-1)^{\tau(12\cdots n)+\tau(q_1q_2\cdots q_n)}a_{1q_1}a_{2q_2}\cdots a_{nq_n}$$

$$=\sum_{q_1q_2\cdots q_n}(-1)^{\tau(q_1q_2\cdots q_n)}a_{1q_1}a_{2q_2}\cdots a_{nq_n}$$

This is because any n order arrangement can be transformed into $12\cdots n$ by a finite number of times. Also, because there are $n!$ $n-order$ permutations in total, hence, the new determinant defined in this article is equal to the classical determinant!

This definition can be extended to row subscript as n different natural numbers and column subscript as n different natural numbers. Just arrange these n different natural numbers from small to large, then let them take values $1,2,\cdots,n$ in sequence!

Example 1 Find the determinant of order n

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

Solution The elements a_{ij} of this determinant can be non-zero only in the case $i=j$, so,

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_n \\ q_1q_2\cdots q_n}}(-1)^{\tau(p_1p_2\cdots p_n)+\tau(q_1q_2\cdots q_n)}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_nq_n}$$

$$=\frac{1}{n!}\sum_{p_1p_2\cdots p_n}(-1)^{\tau(p_1p_2\cdots p_n)+\tau(p_1p_2\cdots p_n)}a_{p_1p_1}a_{p_2p_2}\cdots a_{p_np_n}$$

$$=\frac{1}{n!}\sum_{p_1p_2\cdots p_n}a_{p_1p_1}a_{p_2p_2}\cdots a_{p_np_n}=a_{11}a_{22}\cdots a_{nn}$$

Example 2 Find the determinant of order n

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Solution When $p_1 = 1$, q_1 must be 1, otherwise $a_{p_1 q_1} = 0$, when $p_2 = 2$, q_2 must be 2, otherwise $a_{p_2 q_2} = 0$, and so on $p_i = q_i, i = 1, 2, \cdots, n$,. The following is the same as Example 1.

Example 3 Find the determinant of order n

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2\,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix}$$

Solution For this determinant, only the elements on the second diagonal may not be 0, so only it's worth considering $q_i = n+1-p_i, i = 1, 2, \cdots n$. therefore

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2\,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} =$$

$$\frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n}$$

$$= \frac{1}{n!} \sum_{p_1 p_2 \cdots p_n} (-1)^t a_{p_1 n+1-p_1} a_{p_2 n+1-p_2} \cdots a_{p_n n+1-p_n}$$

Because each permutation changes the parity of the arrangement and the multiplication of numbers satisfies the commutative law, and any arrangement of order n can be transformed into $12\ldots n$ in a finite number of times, therefore

$$t = \tau(12\cdots n) + \tau(n(n-1)\cdots 21) = \frac{n(n-1)}{2}$$

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2\,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_{1n} a_{2\,n-1} \cdots a_{n1}$$

Property Determinant is equal to its transpose determinant. This is a direct result of the symmetry definition of determinants, and also is the best application of this new definition of determinant.

The Expansion Theorem of Determinants by Rows (Columns)

A determinant is equal to the sum of the products of its elements and their corresponding algebraic cofactor in any row (column), that is

$$D = a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in} (i = 1, 2, \cdots, n)$$

$$D = a_{1j} A_{1j} + a_{2j} A_{2j} + \cdots + a_{nj} A_{nj} (j = 1, 2, \cdots, n)$$

Proof Prove only for rows, because as long as it holds for rows, the above property immediately deduce that it holds for columns as well!

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$$

$$\frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n}$$

$$= \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots 1 \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots 1 \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i 1} \cdots a_{p_n q_n}$$

$$+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots 2 \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots 2 \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i 2} \cdots a_{p_n q_n}$$

$$+ \cdots$$

$$+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots k \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots k \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i k} \cdots a_{p_n q_n}$$

$$+ \cdots$$

$$+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots n \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots n \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i n} \cdots a_{p_n q_n}$$

Where, the general term,

$$\frac{1}{n!}\sum_{\substack{p_1\cdots p_i\cdots p_n\\ q_1\cdots k\cdots q_n}}(-1)^{\tau(p_1\cdots p_i\cdots p_n)+\tau(q_1\cdots k\cdots q_n)}a_{p_1q_1}\cdots a_{p_ik}\cdots a_{p_nq_n}$$

$$=\frac{1}{n!}\sum_{\substack{p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ kq_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_1}a_{p_ik}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$+\frac{1}{n!}\sum_{\substack{p_1p_ip_2\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1kq_2\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_2}a_{p_1q_1}a_{p_ik}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$+\cdots$$

$$+\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_ip_{i+1}\cdots p_n\\ q_1q_2\cdots q_{i-1}kq_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_ik}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}$$

$$+\cdots$$

$$+\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_np_i\\ q_1q_2\cdots q_{i-1}q_{i+1}\cdots q_nk}}(-1)^{t_n}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}a_{p_ik}$$

where

$$t_1=\tau(p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n)$$
$$+\tau(kq_1q_2\cdots q_{i-1}q_{i+1}\cdots q_n)$$

$$t_s=\tau(p_1\cdots p_i\cdots p_{i-1}p_{i+1}\cdots p_n)$$
$$+\tau(q_1\cdots k\cdots q_{i-1}q_{i+1}\cdots n)$$
$$s=2,\cdots,n,i>1,$$

when $i=1$,the same processing can be applied.

Because each permutation changes the parity of the arrangement and the number s of digits from p_1 to p_i is equal to the number of digits from q_1 to k , therefore

$$(-1)^{t_s}=(-1)^{\tau(p_1\cdots p_i\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots k\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$=(-1)^{\tau(p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(kq_1\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$=(-1)^{p_i-1+k-1}(-1)^{\tau(p_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$=(-1)^{p_i-1+k-1}(-1)^{\tau(p_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$s=1,2\cdots,n$$

So, the all sub-terms are equal, that is,

$$\frac{1}{n!}\sum_{\substack{p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ kq_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_1}a_{p_ik}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$=\frac{1}{n!}\sum_{\substack{p_1p_ip_2\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1kq_2\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_2}a_{p_1q_1}a_{p_ik}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}$$

$$=\cdots=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_ip_{i+1}\cdots p_n\\ q_1q_2\cdots q_{i-1}kq_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_ik}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$=\cdots=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_np_i\\ q_1q_2\cdots q_{i-1}q_{i+1}\cdots q_nk}}(-1)^{t_n}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$\cdot a_{p_ik}$$

$$=\frac{1}{n!}a_{p_ik}(-1)^{p_i+k}\sum_{\substack{p_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}$$

Hence the general term

$$\frac{1}{n!}\sum_{\substack{p_1\cdots p_i\cdots p_n\\ q_1\cdots k\cdots q_n}}(-1)^{\tau(p_1\cdots p_i\cdots p_n)+\tau(q_1\cdots k\cdots q_n)}a_{p_1q_1}\cdots a_{p_ik}\cdots a_{p_nq_n}$$

$$=\frac{n}{n!}a_{p_ik}(-1)^{p_i+k}\sum_{\substack{p_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}$$

$$=\frac{1}{(n-1)!}a_{p_ik}(-1)^{p_i+k}(n-1)!M_{p_ik}=a_{p_ik}A_{p_ik}$$

Where,

$$t_i=\tau(p_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots q_{i-1}q_{i+1}\cdots q_n)$$

Therefore,

$$D=\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\cdots&\cdots&\cdots&\cdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{vmatrix}=$$

$$D=a_{p_i1}A_{p_i1}+a_{p_i2}A_{p_i2}+\cdots+a_{p_in}A_{p_in}\,(i=1,2,\cdots,n)$$

The theorem has been proven!

Reference

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