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Joint Approximations of 2π-Periodic Functions and Values of Some Widths in L_2 Space

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Abstract

The paper is devoted to studying problems in the theory of approximation of periodic classes of functions by trigonometric polynomials in the Hilbert space L2. Exact constants in Jackson-Stechkin type inequalities are obtained for functions $f \in L_2^{(r)}$, whose successive derivatives f(s) (s = 0,1,... ,r) belong to the space L_2. Also, an exact value of simultaneous approximations of a function and its successive derivatives is obtained for certain classes of functions defined by the generalized modulus of continuity of higher orders $\Omega_m(f^{(r)},t)_2$. For the class of functions $W_m^{(r)}(h)$, where $m \in N$, $r \in Z_+$, $h \in (0,3\pi/(4n)]$, satisfying the constraint

$$\left\{\Omega_m^{2/m}(f^{(r)},h)+\frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right\}^{m/2}\le 1,$$

the exact values of the Bernstein, Kolmogorov, linear, Gelfand, and projection n-widths in the L_2 space are calculated.

Keywords: Periodic Function, Best Simultaneous Approximation, Jackson-Stechkin Inequalities,Trigonometric Polynomial, Modulus of Continuity, Widths.

Introduction

Among the extremal problems in the theory of function approximation, the central issue is finding exact constants in inequalities of the Jackson-Stechkin type. Inequalities of this type, broadly defined in any normed space, are understood as inequalities where the best approximation of functions by a finite-dimensional subspace is estimated through certain characteristics of the smoothness of the function itself or its derivatives. The classical modulus of continuity of a function is typically considered as such a characteristic. Recently, in addressing various extremal problems in function approximation theory, different modifications of the classical smoothness characteristic have frequently been employed. This choice is often dictated by the specifics of the problems under consideration and enables the derivation of new meaningful results.

In this study, we focus on the approximation of 2π-periodic functions by trigonometric polynomials in the space L_2. The result obtained in Theorem 1 extends and generalizes a finding by S.B. Vakarchuk and A.N. Shchitov [1], which was proved for the classical modulus of continuity of the m-th order, $\omega_m(f^{(r)},t)$ to the case of the generalized modulus of continuity $\Omega_m(f^{(r)},t)$ for functions $f \in L_2^{(r)}$

On the best Joint Approximation of Functions in L_2 Space

Let N be the set of natural numbers, and define Z+ = N ∪ {0}; Let R+ denote the set of all positive real numbers. The space L2 is defined as the space of measurable and Lebesgue square-summable real 2π-periodic functions f with finite norm equal to

$$\|f\| \overset{def}{=} \|f\|_{L_2} := \left(\frac{1}{\pi}\int_0^{2\pi} |f(x)|^2 dx\right)^{1/2}$$

The set of all possible trigonometric polynomials of order no greater than n−1 is denoted by $T_{2n}-1$. It is well known that for an arbitrary function $f \in L_2$ with a Fourier series expansion given by

$$f(x) = \frac{a_0(f)}{2} + \sum_{k=1}^{\infty} \rho_k(f)\cos\left(kx + \gamma_k\right) \tag{1.1}$$

where the equality is understood in the sense of convergence in the space L_2, the value $E_n-1(f)_2$ of its best polynomial approximation by elements of the subspace $T_n-1 \in T_{2n}-1$ is given by

$$E_{n-1}(f)_2 = \inf_{T_{n-1}(x)\in\mathscr{T}_{2n-1}} \|f - T_{n-1}\| = \|f - S_{n-1}(f)\| = \left\{\sum_{k=n}^{\infty} \rho_k^2(f)\right\}^{1/2}, \tag{1.2}$$

where

$$S_{n-1}(f,x) = \frac{a_0(f)}{2} + \sum_{k=1}^{n-1} \rho_k(f)\cos\left(kx + \gamma_k\right) \tag{1.3}$$

is the partial sum of order n − 1 of the Fourier series of the function f. Here, $\rho^2_k(f) = a^2_k(f) + b^2_k(f)$, for $k \in N$, where $a_k(f)$ and $b_k(f)$ are the cosine and sine Fourier coefficients of the function f.

We denote by $L_2^{(r)}$ $(r \in \mathbb{Z}_+,\ L_2^{(0)} \equiv L_2)$ the set of functions $f \in L_2$ for which the (r − 1)-th order derivatives $f^{(r-1)}$ are absolutely continuous, and the r-th order derivatives $f^{(r)}$ belong to L_2 space.

Let $\Delta^m_h(f)$ denote the norm of the difference of the m-th order of a function f from the space L_2 with step h:

$$\Delta_h^m(f) := \|\Delta_h^m f(\cdot)\| = \left\{\frac{1}{\pi}\int_0^{2\pi}\left|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k} f(x+kh)\right|^2 dx\right\}^{1/2}$$

and equality

$$\omega_m(f,t) \overset{def}{=} \sup\left\{\Delta_h^m(f):\ |h| \le t\right\} \tag{1.4}$$

we define the modulus of continuity of the m-th order of the function $f \in L2$.

Recall that by Jackson-Stechkin type inequalities in any normed space X we mean relations of the form

$$E_{n-1}(f)_X \le \chi n^{-r}\omega_m(f^{(r)}, \tau/n)_X, \quad r \in \mathbb{Z}_+,\ \tau > 0$$

in which the approximation error of an individual function f is estimated through a given smoothness characteristic ω_m of the approximated function f itself or some of its derivatives $f^{(r)} \in X$. This leads to the problem of finding exact constants in the Jackson-Stechkin inequality that relate the values of the best approximations to the average value of the modulus of continuity defined in (1.4).

Recently, in addressing a number of extremal problems in the theory of function approximation, various modifications of the classical modulus of continuity of the m-th order, as defined in (1.4), have been increasingly utilized. The choice of these modifications is driven by the specific conditions of the problems being studied, enabling the derivation of meaningful results that illuminate the essence of the issues at hand.

In this paper, instead of the classical modulus of continuity of the m-th order for the function $f \in L_2$, we will use the following characteristic of smoothness equivalent to (1.4) of the form

$$\Omega_m(f,t)_2 = \left\{\frac{1}{t^m}\int_0^t \cdots \int_0^t \|\Delta_{\overline{h}}^m f(\cdot)\|^2 dh_1 \cdots dh_m\right\}^{1/2} \tag{1.5}$$

where

$t > 0,\ \overline{h} := (h_1, \cdots, h_m);\ \Delta_{\overline{h}}^m = \Delta_{h_1}^1 \circ \cdots \circ \Delta_{h_m}^1;\ \Delta_{h_j}^1 f(x) := f(x+h_j) -$

$f(x),\ j = \overline{1,m},$ and therefore the calculation of the exact constant is also of some interest

$$\mathscr{L}_{m,n,r}(t) := \sup_{\substack{f\in L_2^{(r)}\\ f\neq\text{const}}} \frac{n^r E_{n-1}(f)}{\Omega_m(f^{(r)}, t/n)}, \quad 0 < t \le 2\pi \tag{1.6}$$

in Jackson-Stechkin type inequality

$$E_{n-1}(f) \le \frac{\mathscr{L}_{m,n,r}(t)}{n^r}\Omega_m(f^{(r)}, t/n)$$

It should be noted that in the study of important questions regarding approximation in the metric space L_p $(0 < p < 1)$, the averaged smoothness characteristic of functions of the form (1.5) was previously considered by E.A. Storozhenko, V.G. Krotov, P. Oswald and K.V. Runovsky [2, 3].

However, the detailed properties of the smoothness characteristic (1.5) were thoroughly investigated in the work of S.B. Vakarchuk, M.Sh. Shabozov, and V.I. Zabutnaya [4].

We will say that a function ϕ is a weight function on the interval [0,h] if ϕ is a non-negative measurable function on [0,h] that is summable and not equivalent to zero.

Let us introduce the notation

$$\operatorname{sinc} u := \left\{\frac{\sin u}{u},\ \text{if}\ \ u \neq 0;\ \ 1,\ \text{if}\ \ u = 0\right\} \tag{1.7}$$

Throughout the following, when calculating the upper bound in general relations for all functions $f \in L_2^{(r)}$, we assume that f =6 const.

Let m,n ∈ N, r,s ∈ Z_+ with r ≥ s and let h ∈ (0,π/n]. We will consider the extremal approximation characteristic of type

$$\mathscr{L}_{n,m,r}^{(s)}(h) := \sup_{f\in L_2^{(r)}} \frac{n^s E_{n-1}(f^{(r-s)})}{\left(\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right)^{m/2}}, \quad (1.8)$$

where the modulus of continuity of the m-th order appears both under the integral sign and outside the integral.

Theorem 1.1. Let m,n ∈ N, r,s ∈ Z_+ with r ≥ s, and let h ∈ (0,3π/(4n)]. Then the equality

$$\mathscr{L}_{n,m,r}^{(s)}(h) = \left(\frac{\sqrt{3}}{nh}\right)^m \quad (1.9)$$

holds.

Proof. By virtue of the equality (1.2), we can write

$$E_{n-1}^2(f) - \sum_{k=n}^{\infty}\rho_k^2(f)\,\mathrm{sinc}kh = \sum_{k=n}^{\infty}\rho_k^2(f)\,(1-\mathrm{sinc}kh) =$$
$$= \sum_{k=n}^{\infty}\left(\rho_k^2(f)\right)^{1-1/m}\left(\rho_k^2(f)\,(1-\mathrm{sinc}kh)^m\right)^{1/m}. \quad (1.10)$$

Applying Holder's inequality for sums [5, p.35] to the right-hand side of the equality (1.10), we obtain

$$E_{n-1}^2(f) - \sum_{k=n}^{\infty}\rho_k^2(f)\,\mathrm{sinc}kh \le$$
$$\le \left(\sum_{k=n}^{\infty}\rho_k^2(f)\right)^{1-1/m}\left(\sum_{k=n}^{\infty}\rho_k^2(f)\,(1-\mathrm{sinc}kh)^m\right)^{1/m} \le$$
$$\le \left(\sum_{k=n}^{\infty}\rho_k^2(f)\right)^{1-1/m}\frac{1}{2n^{2r/m}}\left(2^m\sum_{k=n}^{\infty}k^{2r}\rho_k^2(f)\,(1-\mathrm{sinc}kh)^m\right)^{1/m} \le$$
$$\le \left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2n^{2r/m}}\,\Omega_m^{2/m}(f^{(r)},h),$$
$$\le \left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2n^{2r/m}}\,\Omega_m^{2/m}(f^{(r)},h), \quad (1.11)$$

from which the following inequality follows:

$$E_{n-1}^2(f) \le \sum_{k=n}^{\infty}\rho_k^2(f)\,\mathrm{sinc}kh + \left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2n^{2r/m}}\,\Omega_m^{2/m}(f^{(r)},h). \quad (1.12)$$

Multiplying both sides of the inequality (1.12) by t and integrating over the range t from 0 to τ, we get

$$\frac{\tau^2}{2}E_{n-1}^2(f) \le \sum_{k=n}^{\infty}\frac{1-\cos k\tau}{k^2}\rho_k^2(f) + \left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2n^{2r/m}}\int_0^{\tau} t\,\Omega_m^{2/m}(f^{(r)},t)dt. \quad (1.13)$$

We then integrate the resulting inequality over τ from 0 to h. As a result, we arrive at the following inequality

$$\frac{h^3}{6}E_{n-1}^2(f) \le \sum_{k=n}^{\infty}\frac{kh-\sin kh}{k^3}\rho_k^2(f) +$$
$$+\left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2n^{2r/m}}\int_0^h\int_0^{\tau} t\,\Omega_m^{2/m}(f^{(r)},t)\,dtd\tau \quad (1.14)$$

Using the formula for integration by parts, we can express the integral on the right-hand side of the inequality (1.14) in the form

$$\int_0^h\int_0^{\tau} t\,\Omega_m^{2/m}(f^{(r)},t)\,dtd\tau = \int_0^h \tau(h-\tau)\,\Omega_m^{2/m}(f^{(r)},\tau)\,d\tau. \quad (1.15)$$

Given the equality (1.15), the inequality (1.14) can be expressed in the following form

$$\frac{h^3}{6}E_{n-1}^2(f) \le \sum_{k=n}^{\infty}\frac{kh-\sin kh}{k^3}\rho_k^2(f) +$$
$$+\left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2n^{2r/m}}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)\,d\tau. \quad (1.16)$$

Dividing both sides of the inequality (1.16) by h and replacing the number $1/k^2$ (k ≥ n) under the summation sign with $1/n^2$, we get

$$\frac{h^2}{6}E_{n-1}^2(f) \le \frac{1}{n^2}\left(E_{n-1}^2(f) - \sum_{k=n}^{\infty}\rho_k^2(f)\,\mathrm{sinc}kh\right) +$$
$$+\left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2h\,n^{2r/m}}\int_0^h \tau(h-\tau)\,\Omega_m^{2/m}(f^{(r)},\tau)\,d\tau. \quad (1.17)$$

Applying the inequality (1.11) to the expression in parentheses of the first term on the right-hand side (1.17) and performing simple arithmetic operations, we obtain

$$\frac{h^2}{6}E_{n-1}^2(f) \le$$
$$\le \left(E_{n-1}^2(f)\right)^{1-1/m}\frac{1}{2n^{2+2r/m}}\left(\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\,\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right)$$

(1.18)

From the inequality (1.18), it follows for an arbitrary $f \in L_2^{(r)}$ that

$$\frac{n^r E_{n-1}(f)}{\left(\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right)^{m/2}} \le \left(\frac{\sqrt{3}}{nh}\right)^m,$$

or, what is the same

$$E_{n-1}(f) \le \frac{1}{n^r}\left(\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right)^{m/2}\left(\frac{\sqrt{3}}{nh}\right)^m. \quad (1.19)$$

Assuming r = 0 in the inequality (1.19), we will have

$$E_{n-1}(f) \le \left(\Omega_m^{2/m}(f,h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f,\tau)d\tau\right)^{m/2}\left(\frac{\sqrt{3}}{nh}\right)^m \quad (1.20)$$

Since the inequality (1.20) holds for an arbitrary function $f \in L_2^{(r)}$, then, in particular, replacing f in it with $f^{(r)}$, we write

$$E_{n-1}(f^{(r)}) \le \left(\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right)^{m/2}\left(\frac{\sqrt{3}}{nh}\right)^m \quad (1.21)$$

It is well known that for the best approximations of the sequence of derivatives $f^{(s)}$ (for s = 0,1,...,r) of an arbitrary function $f \in L_2^{(r)}$ the following Kolmogorov-type inequality holds [6, c.122-124]

$$E_{n-1}(f^{(r-s)}) \le \Big(E_{n-1}(f)\Big)^{s/r}\Big(E_{n-1}(f^{(r)})\Big)^{1-s/r} \quad (1.22)$$

By substituting the estimates for $E_n-1(f)$ and $E_n-1(f^{(r)})$ from the righthand sides of (1.19) and (1.21) into the right-hand side of inequality (1.22), we obtain the upper estimate for an arbitrary $0 < h \le \pi/n$

$$\frac{n^s E_{n-1}(f^{(r-s)})}{\left(\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right)^{m/2}} \le \left(\frac{\sqrt{3}}{nh}\right)^m \quad (1.23)$$

To obtain a lower bound, consider the function $f_0(x) = \cos nx$ in the set $L_2^{(r)}$, for which

$$E_{n-1}(f_0^{(r-s)}) = n^{r-s}, \quad \Omega_m^{2/m}\left(f_0^{(r)},\tau\right) = 2\left(1-\frac{\sin n\tau}{n\tau}\right)n^{2r/m},$$

$$\left(\Omega_m^{2/m}(f_0^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f_0^{(r)},\tau)d\tau\right)^{m/2} = n^r\left(\frac{nh}{\sqrt{3}}\right)^m \quad (1.24)$$

Where

$$\mathscr{L}_{n,m,r}^{(s)}(h) := \sup_{f\in L_2^{(r)}} \frac{n^s E_{n-1}(f^{(r-s)})}{\left(\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right)^{m/2}} \ge$$

$$\ge \frac{n^s E_{n-1}(f_0^{(r-s)})}{\left(\Omega_m^{2/m}(f_0^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f_0^{(r)},\tau)d\tau\right)^{m/2}} = \left(\frac{\sqrt{3}}{nh}\right)^m. \quad (1.25)$$

By comparing the upper bound (1.23) with the lower bound (1.25), we obtain the required equality (1.9), which completes the proof of Theorem 1.1.

Values of widths of some classes in the Hilbert space L_2

We present the necessary concepts and definitions to formulate the subsequent results. Let S be the unit ball in L2; Λn ⊂ L2 be an n-dimensional subspace; Λn ⊂ L2 be a subspace of codimension n; L : L2 → Λn be a continuous linear operator; L⊥ : L2 → Λn be a continuous linear projection operator and M be a convex centrally symmetric subset of L_2.

The quantities

$$b_n(\mathfrak{M}, L_2) := \sup\{\sup\{\varepsilon > 0;\ \varepsilon S \cap \Lambda_{n+1} \in \mathfrak{M}\} : \Lambda_{n+1} \subset L_2\},$$

$$d_n(\mathfrak{M}, L_2) := \inf\{\sup\{\inf\{\|f-g\|_2 : g \in \Lambda_n\} : f \in \mathfrak{M}\} : \Lambda_n \subset L_2\},$$

$$\delta_n(\mathfrak{M}, L_2) := \inf\{\inf\{\sup\{\|f-\mathscr{L}f\|_2 : f \in \mathfrak{M}\} : \mathscr{L}f \subset \Lambda_n\} : \Lambda_n \subset L_2\},$$

$$d^n(\mathfrak{M}, L_2) := \inf\{\sup\{\|f\|_2 : f \in \mathfrak{M} \cap \Lambda^n\} : \Lambda^n \subset L_2\},$$

$$\Pi_n(\mathfrak{M}, L_2) := \inf\{\inf\{\sup\{\|f-\mathscr{L}^{\perp}f\|_2 : f \in \mathfrak{M}\} : \mathscr{L}^{\perp}f \subset \Lambda_n\} : \Lambda_n \subset L_2\},$$

are called, respectively, the Bernstein, Kolmogorov, linear, Gelfand and projection n-widths of the set M in L_2. Between the specified n-widths in the Hilbert space L_2, the following relations hold [7, 8]

$$b_n(\mathfrak{M}, L_2) \le d_n(\mathfrak{M}, L_2) \le d^n(\mathfrak{M}, L_2) \le \delta_n(\mathfrak{M}, L_2) = \Pi_n(\mathfrak{M}, L_2) \quad (2.1)$$

By the symbol $W_m^{(r)}(h) := W^{(r)}(\Omega_m, h)$ we denote the set of functions $f \in L_2^{(r)}$, where m ∈ N, r ∈ Z_+ and for any h ∈ (0,3π/(4n)], satisfying the following condition

$$\left\{\Omega_m^{2/m}(f^{(r)},h) + \frac{n^2}{h}\int_0^h \tau(h-\tau)\Omega_m^{2/m}(f^{(r)},\tau)d\tau\right\}^{m/2} \le 1$$

Let's add more

$$E_{n-1}\big(W_m^{(r)}(h)\big) := \sup\left\{E_{n-1}(f) :\ f \in W_m^{(r)}(h)\right\}$$

Following the works , we denote by t∗ the value of the argument t from the interval (0,∞) of the function sincnt, at which it reaches its minimum value [9- 11]. Clearly, t∗ is the smallest positive root of the equation

$$t - \operatorname{tg} t = 0 \ (4{,}49 < t_* < 4{,}51).$$

At the same time, we also believe that

$$\big(1-\operatorname{sinc} nt\big)_* := \begin{cases} 1-\operatorname{sinc} nt, & \text{if } \ 0 < t \le t_*, \\ 1-\operatorname{sinc} nt_*, & \text{if } \ t \ge t_*. \end{cases} \quad (2.2)$$

The following holds true

Theorem 2.1. Let m,n ∈ N and r ∈ Z_+. Then, for all h ∈ (0,3π/(4n)], the following equalities hold

$$\lambda_{2n}\big(W_m^{(r)}(h), L_2\big) = \lambda_{2n-1}\big(W_m^{(r)}(h), L_2\big) = E_{n-1}\big(W_m^{(r)}(h)\big) = \left(\frac{\sqrt{3}}{nh}\right)^m \frac{1}{n^r}, \quad (2.3)$$

where $\lambda_n(\cdot)$ denotes any of the n-widths mentioned above.

Proof. Using the definition of the class $W_m^{(r)}(h)$, from the inequality (1.19), for any h ∈ (0,3π/(4n)] we write down the upper bound for all n widths under consideration

$$\lambda_{2n-1}\big(W_m^{(r)}(h), L_2\big) \le d_{2n-1}\big(W_m^{(r)}(h), L_2\big) \le \quad (2.4)$$

$$\le \sup\left\{E_{n-1}(f) :\ f \in W_m^{(r)}(h)\right\} \le \left(\frac{\sqrt{3}}{nh}\right)^m \frac{1}{n^r}$$

To obtain a lower bound for the Bernstein n-width of the class $W_m^{(r)}(h)$, we consider the (2n + 1)-dimensional ball of polynomials

$$\sigma_{2n+1} = \left\{ T_n(x): \ \|T_n\| \le \left(\frac{\sqrt{3}}{nh}\right)^m \frac{1}{n^r} \right\}$$

and show that the inclusion $\sigma_{2n+1} \subset W_m^{(r)}(h)$ holds. To do this, we need to prove that for an arbitrary trigonometric polynomial $T_n \in \sigma_{2n+1}$, the following inequality holds

$$\left\{ \Omega_m^{2/m}(T_n^{(r)}, h) + \frac{n^2}{h} \int_0^h \tau(h-\tau)\Omega_m^{2/m}(T_n^{(r)}, \tau)d\tau \right\}^{m/2} \le 1. \tag{2.5}$$

Indeed, since for any natural number $1 \le k \le n$ and for $h \in (0,2\pi]$, the following inequality holds

$$1 - \operatorname{sinc}(kh) \le \left(1 - \operatorname{sinc}(nh)\right)_*, \tag{2.6}$$

then, using (2.6), for an arbitrary polynomial $T_n \in T_{2n+1}$, we obtain

$$\Omega_m^2(T_n^{(r)}, \tau) = 2^m \sum_{k=1}^n k^{2r} \rho_k^2(T_n)\left(1 - \operatorname{sinc}(k\tau)\right)^m \le$$

$$\le 2^m n^{2r}\left(1 - \operatorname{sinc}(n\tau)\right)_*^m \sum_{k=1}^n \rho_k^2(T_n) = 2^m n^{2r}\left(1 - \operatorname{sinc}(n\tau)\right)_*^m \|T_n\|^2,$$

or, what is the same,

$$\Omega_m\left(T_n^{(r)}, \tau\right) \le 2^{m/2} n^r \left(1 - \operatorname{sinc} n\tau\right)_*^{m/2} \|T_n\|. \tag{2.7}$$

Using the inequality (2.7) and the relation (2.2), we have

$$\left\{ \Omega_m^{2/m}(T_n^{(r)}, h) + \frac{n^2}{h} \int_0^h \tau(h-\tau)\Omega_m^{2/m}(T_n^{(r)}, \tau)d\tau \right\}^{m/2} \le$$

$$\le \left\{ 2n^{2r/m}(1 - \operatorname{sinc} nh)\|T_n\|^{2/m} + \right.$$

$$\left. + \frac{n^2}{h} \cdot 2n^{2r/m}\|T_n\|^{2/m} \int_0^h \tau(h-\tau)(1 - \operatorname{sinc} n\tau)d\tau \right\}^{m/2} =$$

$$= 2^{m/2} n^r \|T_n\| \left\{ (1 - \operatorname{sinc} nh) + \frac{n^2}{h} \int_0^h \tau(h-\tau)(1 - \operatorname{sinc} n\tau)d\tau \right\}^{m/2} \le$$

$$\le \left(\frac{\sqrt{3}}{nh}\right)^m \left\{ 2(1 - \operatorname{sinc} nh) + \frac{(nh)^2}{3} - 2(1 - \operatorname{sinc} nh) \right\}^{m/2} =$$

$$= \left(\frac{\sqrt{3}}{nh}\right)^m \cdot \left(\frac{nh}{\sqrt{3}}\right)^m = 1$$

and thus, the inequality (2.5) is proved. This implies that the ball σ_{2n+1} belongs to the class $W_m^{(r)}(h)$. Therefore, by the definition of the Bernstein n-width, we can express the lower bound for all n-widths as follows

$$\lambda_{2n-1}\left(W_m^{(r)}(h), L_2\right) \ge \lambda_{2n}\left(W_m^{(r)}(h), L_2\right) \ge$$

$$\ge b_{2n}\left(W_m^{(r)}(h), L_2\right) \ge b_{2n}\left(\sigma_{2n+1}, L_2\right) \ge \left(\frac{\sqrt{3}}{nh}\right)^m \frac{1}{n^r} \tag{2.8}$$

By virtue of the inequalities between the n-widths (2.1) and the comparison of the inequalities (2.4) and (2.8), we obtain the assertion of Theorem 2.1.

Consequence 2.1. From the equality (2.3), for $h = \pi/(2n)$, we have

$$\lambda_{2n}\left(W_m^{(r)}(\pi/(2n)), L_2\right) = \lambda_{2n-1}\left(W_m^{(r)}(\pi/(2n)), L_2\right) =$$

$$= E_{n-1}\left(W_m^{(r)}(\pi/(2n))\right) = \left(\frac{2\sqrt{3}}{\pi}\right)^m \frac{1}{n^r},$$

where $\lambda_n(\cdot)$ denotes any of the n-widths listed above.

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