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Inner Point and Optimization Approach to Constrained Break Even Points

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Abstract
The break even points play important role in business analysis and industrial management. Traditional break even analysis is used when a company is trying to determine what single level of sales, prices, and costs is necessary to reach zero profit. We extend a traditional break even points by introducing a new notion of constrained break even points with respect to parameters. In this case, a traditional method of finding break even points may fail. For this purpose, for finding constrained break even points, we propose inner point method and optimization approach.

Inner point method finds relative interior points of a set for the constrained break even points with respect to volume while optimization methods deal with feasible points of the set. For finding feasible points of the set of constrained break even points, convex minimization and convex maximization algorithms are used. We show that global minimum, local maximum, and stationary points of both problems are the constrained break even points. The proposed approaches are illustrated on some examples providing numerical results.

Keywords: Management Decision-Making, Break Even Points, Set of Constrained Break Even Points, Inner Point Method, Convex Minimization, Convex Maximization.

Introduction

The break even analysis, as a part of cost-volume-profit analysis of business industry, provides management with important information about the relationship between costs, product volume, and profits. Break even analysis is based on the assumption that all costs can be classified into fixed and variable costs. Fixed costs are constant over the range of the analysis while variable costs are proportional to volume. Break even analysis is used to determine the level of sales which is required to recover all costs incurred during the period. In other words, the break even point is the level at which cost and revenue are equal [1].

If sales fall below the break even point, losses will be incurred. Management must determine the break even point to compute the margin of safety, which indicates how much sales may decrease from the targeted level before the company will incur losses. The objective of break even analysis is to determine the volume of sales to achieve a zero profit. There are many works devoted to break even analysis, but most of them deal with one type of product case. However, there is a very simple weighted average contribution margin method developed for multi-product case. It was shown that, break even analysis is one of the most important tools for management in decision making [2 -9].

In the break even analysis has been applied in all businesses and any industry, whether large or small. Break even analysis for the liquefied natural gas plants has been done in [10-13]. The research done in recommended that block industries should use break even analysis for profit maximization purpose. To improve the efficiency of reusable products and obtain an environmental benefit, break even analysis has been used in [15, 16].

Break even analysis for milk production of selected EU countries was done in. In break even analysis in the battery production has been analysed. Uncertainty in parameters, based on uncertainty in the prices, and a risk-return analysis have been examined from a view point of portfolio optimization in taking into account the existing literature on break even analysis, it seems that less attention has been paid to constrained break even points which arise naturally from the traditional break even analysis by imposing constraints on parameters of cost volume-profit analysis [11, 17-19].

To fulfil this gap, we propose a new mathematical methodology called inner point and optimization methods. On the other hand, we continue recent research done in [20] on profitability analysis of business. In this paper, we define constrained break even points with respect to volume. The proposed approach was illustrated on some examples.

The paper is organized as follows. A new concept of sets of constrained break even points has been defined in Chapter 2. Chapter 3 is devoted to inner point methods for finding constrained break even points. Convex minimization approach to break even analysis has been given in Chapter 4. Convex maximization approach to break even analysis has been considered in Chapter 5. The numerical implementation of the proposed approaches has been illustrated in Chapter 6.

A Set of Constrained Break Even Points

The total profit of a company for a multi-product case can be written as

$$\pi = \sum_{j=1}^{n} p_j x_j - \sum_{j=1}^{n} c_j x_j - F \qquad (1)$$

where, π-total profit, pj -price per unit of j-th product, xj -quantity of product sold, cj -variable cost per unit of j-th product, F-total fixed cost, pj > cj. By definition of break even points . π = 0. (2)

Define a set of break even points with respect to volume as follows

$$B_x = \left\{ x \in R^n \; | \sum_{j=1}^{n} (p_j - c_j) x_j = F, \; x_j \geq 0, \; j = 1, \ldots, n \right\} \qquad (3)$$

where, pj, cj, j = 1, . . ., n and F are fixed. In order to find break even points analytically, we construct points.

$$A_j(0, 0, \ldots, \frac{F}{p_j - c_j}, \ldots, 0), j = 1, \ldots, n$$

such that Aj Ɛ Bx.

Define the set of convex combinations of Aj in the following.

$$C = \left\{ y \in R^n \; | y = \sum_{j=1}^{n} \alpha_j A_j, \; \sum_{j=1}^{n} \alpha_j = 1, \; \alpha_j \geq 0, j = 1, \ldots, n \right\}. \qquad (4)$$

Lemma 2.1. Bx = C

Proof. We can easily check that $C \subset B_x$. Indeed, take any point $y \in C$ such that $y = \sum_{j=1}^{n} \alpha_j A_j$, $\sum_{j=1}^{n} \alpha_j = 1$, $\alpha_j \geq 0$ then we compute

$$\sum_{j=1}^{n} (p_j - c_j) y_j = \sum_{j=1}^{n} (p_j - c_j) \frac{\alpha_j F}{p_j - c_j} = F$$

which shows C ⊂ Bx. Inverse conclusion Bx ⊂ C is obvious. Lemma 2.1 allows us to find break even points as many as possible as convex combination's of points Aj, j = 1, . . ., n. Similarly, we introduce the sets of break even points with respect to price and cost. Let us introduce the set of break even points with respect to price p

$$B_p = \left\{ p \in R^n \; | \sum_{j=1}^{n} (p_j - c_j) x_j = F, \; p_j \geq 0, \; j = 1, \ldots, n \right\}$$

where, $x_j, c_j, j = 1, \ldots, n$ and F are fixed.

Introduce the set of break even points with respect to cost c

$$B_c = \left\{ c \in R^n \; | \sum_{j=1}^{n} (p_j - c_j) x_j = F, \; c_j \geq 0, \; j = 1, \ldots, n \right\}$$

where, $x_j, p_j, j = 1, \ldots, n$ and F are fixed.

We introduce the set Dx, called constrained break even points with respect to volume, defined as follows:

$$D_x = \left\{ x \in R^n \mid \sum_{j=1}^{n} (p_j - c_j) x_j = F, \; x_j^{min} \leq x_j \leq x_j^{max}, \; j = 1, \ldots, n \right\} \qquad (5)$$

where, x_j^{min}, x_j^{max}-minimum and maximum capacity volumes for j-th product, $j = 1, \ldots, n$, and $p_j > c_j$, $x_j^{max} \leq \frac{F}{p_j - c_j}$, $D_x \neq \varnothing$.

Now assume that average volumes of products ¯xj, j = 1, . . ., n and fixed cost are given. The set of constrained break even points with respect to price is defined as follows:

$$D_p = \left\{ p \in R^n \;\; | \;\; \sum_{j=1}^{n} (p_j - c_j) \bar{x}_j = F, \; p_j^{min} \leq p_j \leq p_j^{max}, \; j = 1, \ldots, n \right\}.$$

Let the average volumes xj and prices pj of products be given. Define the set of constrained break even points with respect to variable cost as:

$$D_c = \left\{ c \in R^n \; | \sum_{j=1}^{n} (\bar{p}_j - c_j) \bar{x}_j = F, \; c_j^{min} \leq c_j \leq c_j^{max}, \; j = 1, \ldots, n \right\}.$$

In next sections, we restrict ourselves to the set of constrained break even points.

Inner point approach

Problem of finding feasible points of Dx is called constrained break even problem.

$$D_x = \left\{ x \in R^n \mid \sum_{j=1}^{n} (p_j - c_j) x_j = F, \; x_j^{min} \leq x_j \leq x_j^{max}, \; j = 1, \ldots, n \right\} \qquad (6)$$

Introduce the vectors.

$$p - c = (p_1 - c_1, p_2 - c_2, \ldots, p_n - c_n)$$

$$x = (x_1, x_2, \ldots, x_n)$$
$$x^{min} = (x_1^{min}, x_2^{min}, \ldots, x_n^{min})$$
$$x^{max} = (x_1^{max}, x_2^{max}, \ldots, x_n^{max}).$$

Then D_x can be written as

$$D_x = \{x \in R^n \mid \langle p, x \rangle = F, \; x^{min} \leq x \leq x^{max}\},$$

here $\langle , \rangle$ denotes the scalar product of two vectors in R^n.

Note that if the following conditions are satisfied:

$$\langle p - c, x^{min} \rangle > F \qquad (7)$$

$$\langle p - c, x^{max} \rangle > F \qquad (8)$$

then $D_x = \varnothing$. Indeed, if $D_x \neq \varnothing$ there exists a point $\tilde{x}$ such that

$$\langle p, \tilde{x} \rangle = F, \quad x^{min} \leq \tilde{x} \leqslant x^{max}. \qquad (9)$$

On the other hand, $\tilde{x}$ can be represented as

$$\tilde{x} = \alpha x^{min} + (1 - \alpha) x^{max}, \quad 0 \leq \alpha \leq 1.$$

Now taking into account [7] and [8], we have

$$\langle p-c,\tilde{x}\rangle = \alpha\langle p-c,x^{min}\rangle + (1-\alpha)\langle p-c,x^{max}\rangle > \alpha F + (1-\alpha)F = F$$

which contradicts [9]. Thus, we assume that.

$$\begin{cases}\langle p-c,x^{min}\rangle < F\\ \langle p-c,x^{max}\rangle > F\end{cases} \qquad (10)$$

which guarantees non-emptiness of the set Dx.
Condition (10) is equivalent to

$$\langle p-c,x^{min}\rangle < F < \langle p-c,x^{max}\rangle. \qquad (11)$$

Consequently, we have

$$\langle p-c,x^{max}-x^{min}\rangle > 0.$$

Now we show how to find a constrained break even point in D_x. In order to do that we construct a point y as

$$y = \alpha x^{max} + (1-\alpha)x^{min},\ 0 \le \alpha \le 1$$

which is

$$y = \alpha(x^{max} - x^{min}) - x^{min}.$$

Find a value of α from the condition.

$$\langle p-c,y\rangle = F.$$

We rewrite this equation and find α.

$$\langle p-c,\alpha(x^{max}-x^{min}) + x^{min}\rangle = F$$

$$\alpha\langle p-c,x^{max}-x^{min}\rangle = F - \langle p-c,x^{min}\rangle$$

$$\bar{\alpha} = \frac{F - \langle p-c,x^{min}\rangle}{\langle p-c,x^{max}-x^{min}\rangle}.$$

By condition (10), we have $\bar{\alpha} > 0$.
Now, we show that $\bar{\alpha} < 1$. In fact, it can be checked that

$$\bar{\alpha} = \frac{F - \langle p-c,x^{min}\rangle}{\langle p-c,x^{max}-x^{min}\rangle} < 1$$

Indeed

$$F - \langle p-c,x^{min}\rangle < \langle p-c,x^{max}\rangle - \langle p-c,x^{min}\rangle$$

Taking into account

$$\langle p-c,x^{max}\rangle > F$$

we obtain $\bar{\alpha} < 1$.
Thus,

$$y = \bar{\alpha}(x^{max} - x^{min}) + x^{min} \in D_x.$$

Now we define feasible points of the set of constrained break even points with respect to volume. We can easily see that for a given $h \in R^n$ and $\varepsilon \in R$ such that $\langle p-c,h\rangle = 0$ a point z defined as $z = y + \varepsilon h$ satisfies $\langle p-c,z\rangle = F$.
Indeed,

$$\langle p-c,y+\varepsilon h\rangle = \langle p-c,y\rangle + \varepsilon\langle p-c,h\rangle = F.$$

Parameter ε is defined from the condition $z \in D_x$ and

$$x^{min} \le y + \varepsilon h \le x^{max},$$

equivalently,

$$\begin{aligned}&x_i^{min} \le y_i + \varepsilon h_i \le x_i^{max},\quad i = 1,2\ldots,n,\\ &x_i^{min} - y_i \le \varepsilon h_i \le x_i^{max} - y_i,\quad i = 1,2\ldots,n.\end{aligned} \qquad (13)$$

For instance, if we take h as

$$h = (1,1,\ldots 1,h_n),$$

then hn can be found from condition

$$\langle p-c,h\rangle = \sum_{i=1}^{n-1}(p_i - c_i)h_i + (p_n - c_n)h_n = 0.$$

Hence, we find h_n as

$$h_n = -\frac{\sum_{i=1}^{n-1}(p_i - c_i)}{p_n - c_n} < 0.$$

Now it is clear that a point $z = y + \varepsilon h$ is a break-even point with ε such that

$$\begin{cases}x_i^{min} - y_i \le \varepsilon \le x_i^{max} - y_i, \quad i = 1,2,\ldots,n-1\\ x_n^{min} - y_i \le -\varepsilon\dfrac{\sum_{i=1}^{n-1}(p_i - c_i)}{p_n - c_n} \le x_n^{max} - y_n\end{cases} \qquad (14)$$

From this expression parameter ε can be chosen as follows:

$$\max\{\max_{1\le i\le n-1}\{x_i^{max} - y_i\}, \frac{(y_n - x_n^{max})(p_n - c_n)}{\sum_{i=1}^{n-1}(p_i - c_i)}\} \le \varepsilon \qquad (15)$$

$$\varepsilon \le \min\{\min_{1\le i\le n-1}\{x_i^{max} - y_i\}, \frac{(y_n - x_n^{min})(p_n - c_n)}{\sum_{i=1}^{n-1}(p_i - c_i)}\} \qquad (16)$$

In particular, for $h = (1,0,\ldots 0,h_n^0)$, we have

$$\langle p-c,h\rangle = (p_1 - c_1) + (p_n - c_n)h_n^0 = 0.$$

Hence, we find h_n:

$$h_n^0 = \frac{c_1 - p_1}{p_n - c_n}$$

Then parameter ε can be chosen from condition (13):

$$\begin{cases}x_1^{min} - y_1 \le \varepsilon \le x_1^{max} - y_1\\ x_n^{min} - y_n \le \varepsilon h_n^0 \le x_n^{max} - y_n\end{cases} \qquad (17)$$

Since $h_n^0 < 0$ and $x_n^{max} - y_n \ge 0$, then the last inequalities can be written:

$$\begin{cases}x_1^{min} - y_1 \le \varepsilon \le x_1^{max} - y_1\\ \varepsilon \le \dfrac{y_n - x_n^{min}}{(p_n - c_n)}\end{cases} \qquad (18)$$

Hence, we have

$$x_1^{min} - y_1 \le \varepsilon \le \min\{x_1^{max} - y_1, \frac{y_n - x_n^{min}}{(p_n - c_n)}\}. \qquad (19)$$

A procedure of finding constrained break-even points is given in the following algorithm.

Algorithm -CBEP

Step 1: Compute a value of α by formula [12]

$$\alpha = \frac{F - \langle p-c,x^{min}\rangle}{\langle p-c,x^{max}-x^{min}\rangle}.$$

Step 2: Construct a point y as

$$y = \alpha(x^{max} - x^{min}) + x^{min}.$$

Step 3: Choose an arbitrary vector h0 Rn such that.
$\langle p-c,h^0\rangle = 0$ by solving the equation

$$(p_1 - c_1)h_1^0 + (p_2 - c_2)h_2^0 + \cdots + (p_n - c_n)h_n^0 = 0.$$

Step 4: Choose a parameter ε from the following conditions.

$$x_i^{min} - y_i \le \varepsilon h_i^0 \le x_i^{max} - y_i,\quad i = 1,2,\ldots n.$$

For $h_i^0 > 0$:

$$\frac{x_i^{min} - y_i}{h_i^0} \le \varepsilon \le \frac{x_i^{max} - y_i}{h_i^0},\ i = 1,2,\ldots,n,$$

For $h_i^0 < 0$:

$$\frac{x_i^{max} - y_i}{h_i^0} \le \varepsilon \le \frac{x_i^{min} - y_i}{h_i^0},\ i = 1,2,\ldots n.$$

Combining the above inequalities, we find a range of parameters of ε:

$$\max\{\max_{h_i^0>0}\{\frac{x_i^{min} - y_i}{h_i^0}\}, \max_{h_i^0<0}\{\frac{x_i^{max} - y_i}{h_i^0}\}\} \le \varepsilon$$

$$\varepsilon \le \min\{\min_{h_i^0>0}\{\frac{x_i^{max} - y_i}{h_i^0}\}, \min_{h_i^0}\{\frac{x_i^{min} - y_i}{h_i^0}\}\}$$

Step 5: Compute a break-even point z:

$$z = y + \varepsilon h^0.$$

Lemma 3.1. The points {z} generated by Algorithm-CBEP for any h^0 Rn are constrained break even points, that is z Dx for all h^0 Rn.

The proof follows from formulas [12-19].
Note that points {z} generated by Algorithm-CBEP are also relative interior points or inner points of the set Dx.

The main difference between traditional break even analysis and proposed inner point approach is the following. In traditional approach a single break even point can be found satisfying the equation π = 0 for fixed cost and prices. The break even point in the literature is defined by the ratio of fixed cost and sum of unit contribution margins as follows [3]

$$\tilde{x}_j = \frac{F}{\sum_{j=1}^{n}(p_j - c_j)}, \; j = 1, \dots, n. \tag{20}$$

Also, its extended form in (4):

$$y = \sum_{j=1}^{n} \frac{\alpha_j F}{(p_j - c_j)}, \qquad \sum_{j=1}^{n} \alpha_j = 1$$

are break even points in a traditional sense, but may not be constrained break even points for certain αj . In inner point approach, we find a set of constrained break even points with respect to not only volumes but also prices and variable costs by Algorithm-CBEP.

Convex Programming Approach

Another way of finding a feasible point in the set of constrained break even points is to solve the following convex programming problem.

$$\min_{x \in D_x} \; f = \|x - u^0\|^2 \tag{21}$$

where, $u^0 \in R^n$ is an arbitrary initial point and

$$D_x = \left\{ x \in R^n \mid \sum_{j=1}^{n}(p_j - c_j)x_j = F, \; x_j^{min} \le x_j \le x_j^{max}, \;\; j = 1, \dots, n \right\}.$$

If we solve problem (21) for different initial points $u^j, \; j = 1, \dots, m$, then its corresponding solutions $x^j, \; j = 1, \dots, m$, are constrained break even points in D_x. That is,

$$\min_{x \in D_x} \|x - u^j\|^2 = \|x^j - u^j\|^2, \; j = 1, \dots, m.$$

It is obvious that the points z_α constructed by

$$z_\alpha = \sum_{j=1}^{m} \alpha_j x^j$$

for any αj such that

$$\sum_{j=1}^{m} \alpha_j = 1, \; \alpha_j \ge 0, j = 1, \dots, m$$

are also the constrained break even points.

Since the set of constrained break even points is compact consisting of linear constraints, for solving problem [21], we use Conditional Gradient Method [5]. Algorithm of Conditional Gradient Method is the following.

Algorithm of Conditional Gradient Method (CGA1)

Step 1: Choose an arbitrary point x^0 Rn. x^k Rn, k = 0

$$\min_{x \in D_x} \langle f'(x^k), x \rangle$$

Let $\bar{x}^k$ be a solution, that is

$$\min_{x \in D_x} \langle f'(x^k), x \rangle = \langle f'(x^k), \bar{x}^k \rangle$$

Step 3: Compute a value of η_k as

$$\eta_k = \langle f'(x^k), \bar{x}^k - x^k \rangle$$

Step 4: If $\eta_k = 0$ then stop, x^k is a solution to problem; Otherwise, go to next step.

Step 5: Consruct a ray for $\alpha \in [0, 1]$:

$$x^k(\alpha) = x^k + \alpha(\bar{x}^k - x^k).$$

Choose α_k from the condition $f(x^k(\alpha)) < f(x^k)$ or

$$\min_{\alpha \in [0,1]} f(x^k(\alpha)) = f(x^k(\alpha_k)).$$

Step 6: Construct a next approximation point x^{k+1}

$$x^{k+1} = x^k + \alpha_k(\bar{x}^k - x^k),$$

set $k := k + 1$, and go to step2.

Note that in order to find a step size αk in Step 5 of the algorithm, first we need to solve unconstrained one-dimensional quadratic minimization problem:

$$\min_{\alpha \in R} f(x^k(\alpha)) = f(x^k(\bar{\alpha})).$$

Since $f(x^k(\alpha)$ is strongly convex quadratic function, its global minimum computed easily as

$$\bar{\alpha} = -\frac{\eta_k}{\|\bar{x}^k - x^k\|^2}.$$

Now, taking into account $\alpha \in [0, 1]$, we conclude that

$$\alpha_k = \min\{1; \bar{\alpha}\}.$$

Theorem 4.1. *[5] The sequence $\{x^k, k = 0, 1, \dots\}$ generated by the algorithm-CGA1 is a minimizing sequence, that is* $\lim_{k \to \infty} f(x^k) = \min_{x \in D_x} f(x)$.
Any limit point of x^k is a solution to problem:

$$\lim_{k \to \infty} x^k = x^* = argmin_{x \in D_x} f(x).$$

Convex Maximization Approach

In order to find constrained break even points in Dx, it is also possible to solve the following convex maximization problem.

$$\max_{x \in D_x} \; Q = \|x - u^0\|^2 \tag{22}$$

where, $u^0 \in R^n$ is an arbitrary initial point.
Unlike problem(21), this problem is nonconvex and has a finite number of local maximum points.

If we solve problem (22) for different initial points $u^j, \; j = 1, \dots, m$, then its corresponding local solutions or stationary points $x^j, \; j = 1, \dots, m$, are constrained break even points in D_x. That is,

$$\max_{x \in D_x} \|x - u^j\|^2 = \|x^j - u^j\|^2, \; j = 1, \dots, m.$$

It is clear that the points z_α constructed by

$$z_\alpha = \sum_{j=1}^{m} \alpha_j x^j$$

for any α_j such that

$$\sum_{j=1}^{m} \alpha_j = 1, \; \alpha_j \ge 0, j = 1, \dots, m$$

are also the constrained break even points.
Since problem [22] is nonconvex, Algorithm of Conditional Gradient Method cannot always guarantee finding global solutions to the convex maximization problem but may provide local

solutions or stationary points for the problem. Then algorithm of conditional gradient method (CGA1) is modified for the problem as follows.

Algorithm of Conditional Gradient Method (CGA2)

Step 1: Choose an arbitrary point $x^0 \in R^n$. $x^k \in R^n, k = 0$

Step 2: Solve a linear programming problem

$$\max_{x \in D_x} \langle Q'(x^k), x \rangle$$

Let $\bar{x}^k$ be a solution, that is

$$\max_{x \in D_x} \langle Q'(x^k), x \rangle = \langle Q'(x^k), \bar{x}^k \rangle$$

Step 3: Compute a value of η_k as

$$\eta_k = \langle Q'(x^k), \bar{x}^k - x^k \rangle$$

Step 4: If $\eta_k = 0$ then stop, x^k is a solution to problem; Otherwise, go to next step.

Step 5: Consruct a ray for $\alpha \in [0, 1]$:

$$x^k(\alpha) = x^k + \alpha(\bar{x}^k - x^k).$$

Choose α_k from the condition $f(x^k(\alpha)) > f(x^k)$ by the bisection method.

Step 6: Construct a next approximation point x^{k+1}

$$x^{k+1} = x^k + \alpha_k(\bar{x}^k - x^k),$$

set $k := k + 1$, and go to step2.

Theorem 5.1. [6] The sequence {xk, k = 0, 1, . . .} generated by the algorithm-CGA2 converges to a stationary point of problem [22], that is

$$\lim_{k \to \infty} \langle Q'(x^k), \bar{x}^k - x^k \rangle = 0.$$

The proposed optimization approaches can be easily extended for the case of joint constraints of products when Dx is given as follows:

$$D_x = \left\{ x \in R^n \mid \sum_{j=1}^{n} (p_j - c_j) x_j = F,\ Ax \le b,\ \ x_j^{min} \le x_j \le x_j^{max},\ \ j = 1, \ldots, n \right\},$$

where A is a matrix of dimension mxn, and b R^m.

Numerical Implementation

In order to illustrate the proposed interior point approach numerically, we use company's parameters such as price, volume, variable cost and fixed cost.

Table 1: Parameters

i	unit price, p_i	unit variable cost, c_i	min.level of x_i	max.level of x_i
1	700.0	280.0	12.0	30.0
2	1,120.0	420.0	8.0	16.0
3	380.0	340.0	15.0	85.0

Consider a company with the fixed cost of F = 14, 000.0 and parameters given in Table 1. Using these parameters, in Table 2, we find constrained break even points with respect to volume by the interior point method for h = (1, 0, h03) with ε computed by formula [17] as $-3.11 \le \varepsilon \le 0.30$.

Table 2: Inner point method

i	values of $\mathcal{E}$	constrained break even points, z_i
1	0.1	(15.21, 9.38, 26.06)
2	0.15	(15.26, 9.38, 25.53)
3	0.16	(15.27, 9.38, 25.43)
4	0.17	(15.28, 9.38, 25.32)
5	0.18	(15.29, 9.38, 25.22)
6	0.19	(15.30, 9.38, 25.11)
7	0.20	(15.31, 9.38, 25.01)
8	0.21	(15.32, 9.38, 24.90)
9	0.22	(15.33, 9.38, 24.80)
10	0.23	(15.34, 9.38, 24.69)
11	0.24	(15.35, 9.38, 24.59)
12	0.25	(15.36, 9.38, 24.48)
13	0.26	(15.37, 9.38, 24.38)
14	0.27	(15.38, 9.38, 24.27)
15	0.30	(15.41, 9.38, 23.96)

In Table 3, we find constrained break even points with respect to volume by the inner point method for h = (1, 1, h03) with ε computed by formula [17] as
$-1.38 \leq \varepsilon \leq 0.43.$

In Table 4, we provide constrained break even points found by convex minimization method for different initial points of x0.
In Table 5, we provide constrained break even points found by convex maximization algorithm for different initial points of x0.

Table 3: Inner point method

i	values of ϵ	constrained break even points, z_i
1	-1	(14.11, 8.38, 55.11)
2	-0.50	(14.61, 8.88, 41.11)
3	0.00	(15.11, 9.38, 27.11)
4	0.10	(15.27, 9.54, 24.31)
5	0.15	(15.26, 9.53, 22.91)
6	0.16	(15.27, 9.54, 22.63)
7	0.17	(15.28, 9.55, 22.35)
8	0.18	(15.29, 9.56, 22.07)
9	0.19	(15.30, 9.57, 21.79)
10	0.20	(15.31, 9.58, 21.51)
11	0.21	(15.32, 9.59, 21.23)
12	0.22	(15.33, 9.60, 20.95)
13	0.23	(15.34, 9.61, 20.67)
14	0.24	(15.35, 9.62, 20.39)
15	0.25	(15.36, 9.63, 20.11)
16	0.26	(15.37, 9.64, 19.83)
17	0.27	(15.38, 9.65, 19.55)
18	0.30	(15.41, 9.68, 18.71)
19	0.40	(15.51, 9.78, 15.91)

Table 4: Convex minimization

i	initial points, x0	constrained break even points, zi
1	(11.09, 7.60, 68.04)	(13.13, 8.00, 72.10)
2	10.00, 6.32, 67.10)	(12.00, 8.62, 73.10)
3	(9.54.00, 7.12, 71.33)	(12.00, 8.00, 84.00)
4	(11.24, 9.19, 13.73)	(13.13, 11.22, 15.73)
5	(10.34, 6.12, 71.54)	(12.07, 8.00, 83.29)
6	(14.16, 9.67, 16.76)	(12.86, 11.43, 15.00)
7	(11.83, 6.79, 68.25)	(12.64, 8.00, 77.33)
8	(11.00, 14.00, 83.00)	(12.00, 8.67, 72.29)
9	(17.00, 10.00, 45.00)	(14.72, 8.68, 43.48)
10	(41.00, 23.00, 63.00)	(14.34, 8.00, 59.48)
11	(8.00, 4.00, 13.00)	(15.82, 9.59, 15.96)
12	(36.00, 37.00, 22.00)	(17.95, 8.00, 21.57)
13	(15.00, 19.00, 42.00)	(12.77, 10.03, 40.33)
14	(11.00, 9.00, 7.00)	(12.00, 11.94, 15.00)
15	(31.00, 45.00, 13.00)	(18.57, 8.00, 15.00)

Table 5 Conex maximization

i	initial points, x0	constrained break even points, zi
1	(41, 23, 63)	(12, 8, 84)
2	(8, 4, 13)	(12, 11.94, 15)
3	(36, 37, 22)	(18.57 8, 15)
4	(11, 9, 7)	(12, 11.94 15)
5	(31, 45, 13)	(18.57, 8, 15)
6	(14.16, 9.67, 16.76)	(12.86, 11.43, 15.00)

In Table 7, we present break even points found by convex minimization of a com- pany with 50 products and fixed cost of F=14,000.0 based on its parameters given in Table 6.

Table 6: Parameters

i	unit price, pi	unit variable cost, ci	min.level of xi	max.level of xi
1	700.0	280.0	0.0	33.3
2	1,120.0	420.0	0.0	20.0
3	380.0	340.0	0.0	350.0
4	250.0	170.0	0.0	175.0
5	500.0	290.0	0.0	87.5
6	880.0	650.0	0.0	60.8
7	1120.0	840.0	0.0	50.0
8	150.0	80.0	0.0	200.0
9	180.0	60.0	0.0	116.6
10	480.0	450.0	0.0	466.6
11	730.0	655.0	0.0	186.6
12	220.0	160.0	0.0	233.3
13	533.0	483.0	0.0	280.0
14	90.0	70.0	0.0	700.0
15	85.0	75.0	0.0	1400.0
16	133.0	108.0	0.0	560.0
17	155.0	133.0	0.0	636.3
18	55.0	40.0	0.0	933.3
19	600.0	330.0	0.0	51.8
20	905.0	698.0	0.0	67.6
21	770.0	600.0	0.0	82.5
22	78.0	58.0	0.0	700.0
23	95.0	65.0	0.0	466.6
24	1010.0	805.0	0.0	68.2
25	250.0	150.0	0.0	140.0
26	365.0	175.0	0.0	76.6
27	540.0	366.0	0.0	80.4
28	830.0	540.0	0.0	48.2
29	770.0	400.0	0.0	37.8
30	915.0	800.0	0.0	121.7
31	335.0	223.0	0.0	125.0
32	466.0	366.0	0.0	140.0
33	1210.0	1010.0	0.0	70.0
34	1280.0	996.0	0.0	49.3
35	169.0	90.0	0.0	200.0
36	180.0	90.0	0.0	155.5
37	155.0	72.0	0.0	168.6
38	270.0	115.0	0.0	90.3
39	620.0	460.0	0.0	87.5
40	640.0	505.0	0.0	103.7
41	930.0	680.0	0.0	56.0
42	1060.0	990.0	0.0	200.0
43	1,175.0	965.0	0.0	66.6
44	1230.0	890.0	0.0	41.1
45	740.0	450.0	0.0	48.2
46	865.0	663.0	0.0	69.3
47	975.0	820.0	0.0	90.3
48	1085.0	750.0	0.0	41.7
49	355.0	250.0	0.0	133.0
50	410.0	200.0	0.0	66.6

Table 7: Constrained break even points

i	break even point, x1	break even point, x2	break even point, x3	max.level of xi
1	0.00	1.00	1.00	33.3
2	0.00	1.00	1.00	20.0
3	4.42	4.39	4.30	350.0
4	3.85	3.78	3.60	175.0
5	2.71	2.56	2.21	87.5
6	1.71	1.50	1.00	60.8
7	0.99	0.74	1.00	50.0
8	3.99	3.93	3.78	200.0
9	3.28	3.17	2.91	116.6
10	4.57	4.54	4.78	466.6
11	3.92	3.86	3.69	186.6
12	4.14	4.08	3.95	233.3
13	4.28	4.24	4.13	280.0
14	4.71	4.69	4.65	700.0
15	4.85	4.84	4.82	1400.0
16	4.64	4.62	4.56	560.0
17	4.68	4.66	4.61	636.3
18	4.78	4.77	4.73	933.3
19	1.14	0.89	1.00	51.8
20	2.04	1.85	1.39	67.6
21	2.57	2.41	2.04	82.5
22	4.71	4.69	4.65	700.0
23	4.57	4.54	4.47	466.6
24	2.07	1.88	1.43	68.2
25	3.57	3.48	3.26	140.0
26	2.28	2.11	1.69	76.6
27	2.51	2.35	1.97	80.4
28	0.85	0.59	1.00	48.2
29	0.00	0.00	1.00	37.8
30	3.35	3.25	2.99	121.7
31	3.39	3.29	3.05	125.0
32	3.57	3.48	3.26	140.0
33	2.14	1.96	1.52	70.0
34	0.94	0.68	1.00	49.3
35	3.99	3.93	3.78	200.0
36	3.71	3.63	3.43	155.5
37	3.81	3.73	3.55	168.6
38	2.78	2.64	2.30	90.3
39	2.71	2.56	2.21	87.5
40	3.07	2.94	2.65	103.7
41	1.42	1.20	1.00	56.0
42	3.99	3.93	3.78	200.0
43	1.99	1.80	1.34	66.6
44	0.14	0.00	1.00	41.1
45	0.85	0.59	1.00	48.2
46	2.11	1.93	1.48	69.3
47	2.78	2.64	2.30	90.3
48	0.21	0.00	1.00	41.7
49	3.49	3.40	3.17	133.0
50	1.99	1.80	1.34	66.6

Note that in computational experiments break even points found by linear programming and convex maximization algorithm were only the vertices of the set Dx:

$$u^j = (0, 0, \ldots, \frac{F}{p_j - c_j}, \ldots, 0), j = 1, \ldots, 50.$$

Conclusion

In this paper, the main part of traditional cost-volume-profit analysis, break even points have been extended by introducing a new notion of constrained break points for business with respect to volume. The constrained break even points arise when lower and upper constraints are imposed on the volume. In this case, a traditional method for defining break even points may fail even for a single product. We propose a new methodology for finding constrained break even points for multi-product case called inner point and optimization approaches.

The advantage of the proposed approach is that it can handle multi-product case for finding constrained break even points with respect to volume. This helps managers to make rational decisions in profitability of industries. The proposed approaches were tested on some examples providing numerical results obtained on Matlab. Similarly, the proposed approach can be easily extended for the constrained break even points with respect to price as well as a variable cost.

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