On Quantum Entanglement and Nonlocality

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Abstract

Bell inequalities on local hidden variables were used to test particle entanglement given by the quantum mechanics wavefunction. The experiments on correlated photon pairs showed a clear violation of Bell inequalities and agreement with quantum mechanics. This study revealed that the Bell inequalities on normalized density function is true only for photon polarization at individual instances but it does not hold over the entire population in a given system. The Bell inequalities is not an incompatibility criterion for local hidden variables vs quantum mechanics rather it is a criterion for an individual nature vs population dynamics. The study found that the local variables (photon polarization) described each quantity in Bell test and a complete agreement with the experiments where quantum mechanics was incapable to provide as it is a unified system. The paper concludes with great satisfaction that the simultaneous measurement of two correlated photons at a distance was a local cause (photon-filter interaction did not violate the special theory of relativity) and the nonlocality description given by the wavefunction was not physical

Keywords: Quantum Mechanics, Hidden Variables, Bell Theorem, Entanglement, Nonlocality

Graphical Abstract
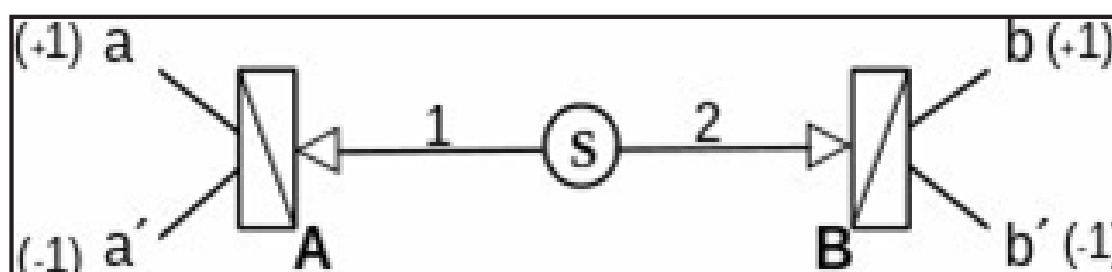
Experimental setup shows that two photons emitted from the same source interacting with two distance filters. In quantum mechanical description, two particles are entangled by two orthogonal wavefunctions (superposition). This violates the special theory of relativity (no information can travel faster than the speed of light). This study shows that quantum mechanics wavefunction is not physical and photon polarization states (local causality) described the lab data completely.

Introduction

Entanglement of two particles emitted from the same source, given by the wavefunction, violates the special theory of relativity. Einstein, Podolsky and Rosen [1] suggested that quantum mechanics is incomplete in describing physical reality. Einstein's acceptable criterion of a reality: "If, without in any way disturbing a system, we can predict with certainty (i.e., with probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity." Following this classical paper, there has been a significant interest in possible hidden variable theories and experimental attempts to address the Einstein Podolsky Rosen paradox (known as the EPR paradox). Clauser et al. [2] quoted Bohr's argument that "no paradox can be derived from the assumption of completeness if one recognizes that quantum mechanics concerns only the interaction of microsystems with the experimental apparatus and not their intrinsic character."

Some arguments have been made to restore new variables to the theory of local causality [3-5]. John Bell derived a theorem stating that any such theory with additional variables will not have complete agreement with quantum mechanics [6]. Clauser and Freedman first conducted Bell tests and drew an affirmative conclusion in favor of quantum mechanics. Aspect et al. [7] conducted similar experiments with more refined settings on correlated photon pairs using two-channel polarizers (analogs of Stern– Gerlach filters on pairs of spin 1/2 particles in a singlet state). Their results showed a clear violation of Bell inequalities (local hidden variables criterion) and a complete agreement with quantum mechanics. A team led by Anton Zeilinger conducted the Cosmic Bell test and arrived at the same conclusion in favor of quantum mechanics. The entanglement and violation of the Bell inequalities in classical optics are now routinely observed. It is an overwhelming consensus that two particles emitted from the same source are entangled in a unified system as described by quantum mechanics.

In the Bell test, there are four measurement quantities (i.e., four elements of realization). The quantities are used to calculate the Bell inequalities. In quantum mechanics, these quantities have no separate realization. This gave us a good reason to formulate local hidden variables (photon polarization) equations for each quantity in Bell test (which was incomplete previously). The hidden variables formulation showed that the correlated photons described each quantity in the Bell test and a complete agreement with the experiments where quantum mechanics was incapable to provide as it is a unified system. The simultaneous measurement of two correlated photons at a distance was a local cause (photon-filter interaction did not violate the special theory of relativity) and the quantum mechanical description given by wavefunction on a unified system is an approximation (not physical). The study then revealed that the Bell inequalities, $-2 \leq S \leq 2$, on normalized density function is true only for any individual photon polarization states but it does not hold true over the entire population in a given system.

This paper is organized into the following sections: Section 2 provided an overview on some perspectives on local causality. This overview includes: quantum wavefunction, classical optics, and photon polarization. Section 3 presented the Bell inequalities and hidden variables equations for four quantities in Bell test. Section 4 provided the hidden variables predictions and compared them with the quantum mechanics. Section 5 compared the Bell inequalities (hidden variables, quantum mechanics and Bell experiments). The Bell experiments here are widely known in the scientific communities and in the general public.

Perspectives on Local Causality

On quantum wavefunction, an argument is that the quantum mechanical wavefunction will never be able to provide a complete picture of a physical system. If we accept the wavefunction as a complete theory, then we have to accept its many strange outcomes, such as action travel faster than light (the violation of special theory of relativity) or Schrödinger cat illustration on wavefunction superposition (alive and dead at the same time). The Schrödinger equation gives a deterministic evolution of the quantum states, but is incapable of describing all quantum properties (such as the states of a hydrogen atom where the wavefunction gives the probability for the electron to be at different locations from the nucleus). In quantum mechanics, the expected value for a physical quantity is obtained by multiplying the corresponding operator by the wavefunction T and its complex conjugate T* as for momentum $\langle p \rangle = \int \Psi * (\hat{p}) \Psi \, dx$, position $\langle x \rangle = \int \Psi * (\hat{x}) \Psi \, dx$ and likewise for other quantities; here, $\hat{p}$ and $\hat{x}$ are the momentum and position operators, respectively. The uncertainties between these two quantities occur naturally in the position-momentum space of the wavefunction. EPR paper illustrated this incomplete description of a physical system given by the quantum mechanical wavefunction. One concluding remark here is that the Schrödinger equation [8-11] was derived (engineered) by satisfying several known conditions of a quantum system (i.e., Planck-Einstein postulate, de Brogie hypothesis, wavefunctions linear superposition), not on the same character of a theory about a natural law, for example energy quanta in the black body radiation by Planck or principle of covariance in the relativity by Einstein.

In Classical Optics, there has been extensive research on optical entanglement and violation of the Bell inequalities. This study reviewed a large volume of related research and found no clear consensus between local causality (such as hidden variables) and nonlocality (as described by quantum mechanics). On quantum science and technology, Heinrich et al. [12] provided a review on extended research on the fundamental principles and practical applications of quantum coherence in nanoscale systems. The authors emphasized the importance of controlling matter and waves at the nanometer scale to increase quantification energies, interaction strengths and coherence times and unlock novel functionalities explicitly enabled by quantum mechanics. On optical experiments and quantum reality, the author Anil Ananthaswamy provided a nice summary of the status of today's experiments on Thomas Young double-slit interference patterns [13]. This review summary included several well-known papers on optical experiments. These experimental papers [14-22] were reviewed here with great interest. For the moment, we are just appreciating the perspective provided by these articles on the modern tools and issues that address some intrinsic nature of particles. A tentative theory of light quanta was first given by de Broglie [23] and a few years later the experimental data on the scattering of electrons by a single crystal of nickel was presented by Davisson and Germer [24]. On the basis of numerical com-

putations of the dynamics of wave–particle duality, Orefice et al. [25] provided a systematic review of mathematical development, starting from de Broglie's and Schrödinger's foundations of wave mechanics in a consistent wave-mechanical extension of classical dynamics.

On photon interaction, a photon will pass through a filter if its electric field is parallel to the filter and will be blocked by a filter oriented perpendicularly. For any other orientation angle, θ, the expected value for the photon will pass is cos θ. The intensity of the transmitted light varies as the square of the cosine of the angle as Malus's law. The photon polarization states will change depending on the orientation of the filter. Such a change in photon states can easily be viewed if photons interact with 2 or 3 separate filters at different orientations. According to the EPR picture of physical reality, each uninterrupted photon along its flight path has a specific polarization state (which is given or altered by the equipment) that can be known without disturbing it. Bell provided an interesting illustration of a spin half particle in a pure spin state vector relative to the observation filter and of the uncertainty of the desired expected value for individual instances. Let us now imagine the polarizing states of all photon pairs produced from the Aspect et al. experiment: a (J=0)-(J=1)-(J=0) cascade in calcium-40 excited by two-photon absorption, with the use of two single-mode lasers [7]. Aspect et al. was able to produce pairs of visible photons (at wavelengths of 551. 3 nm and 422. 7 nm) at a typical rate of 5x10 7 s-1. The flux of photon pairs in the experiment was safely assumed to be an ensemble of all polarization states over the entire 360-degree hemisphere. The quantities of Bell inequalities were subsequently measured, and violations of Bell inequalities were confirmed.

Hidden Variables Formulation
Until now, many Bell test experiments were conducted and came to conclusions in favor of quantum mechanics. Such a conclusion was incomplete as these studies did not have any separate hidden variables predictions. This paper now presents the local hidden variables formulation (previously not known). Follow up sections then provided systematic hidden variables predictions and compared them with quantum mechanics and Bell test experimental data.

Bell Inequalities
Clauser et al. [2] presented a generalization of Bell inequalities by avoiding Bell's experimentally unrealistic restriction for some photon pairs with perfect correlation. Aspect et al. [7] experiments on Bell inequalities using pairs of visible photons emitted in atomic radiative cascades involve the transposition of Einstein–Podolsky–Rosen-Bohm gedanken experiment.
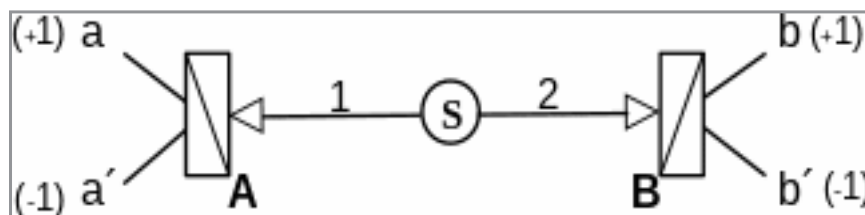
Figure l: Experimental setup similar to the Einstein–Podolsky–Rosen-Bohm gedanken experiment. Two polarizers in two different orientations: a and a' on side A and b and b' on side B. Pairs of photons from a source in a non-factorizing state (similar to two spin-1/2 particles in a singlet state).

John Bell showed that the following expected value of two independent measurements,

$P(\vec{a}\,\vec{b}) = \int A(\vec{a}, \lambda) B(\vec{b}, \lambda)(\lambda)\,d\lambda$, is incompatible with quantum mechanics. Here, $\vec{a}$, $\vec{b}$
are the
are the

orientation vectors of the polarizers, λ, is a unit vector representing the photon polarization states. Bell described the parameter λ, where it stands for any number of variables and the dependencies of A and B
polarizers, or it would have dynamical significance and laws of motion one would envision. By
assuming a normalized density function, $\bar{\rho}$, of an ensemble of the state vectors over the entire 360^0 hemisphere, the generalized Bell inequalities was given as:

$$-2 \leq S \leq 2 \qquad (1)$$

where $S = E(\vec{a}, \vec{b}) - E(\vec{a}, \vec{b}'') + E(\vec{a}', \vec{b}) + E(\vec{a}', \vec{b}'')$, four quantities in the inequalities are the expected values of photons at two observation sites (A, B). Let us use the symbol "p" when a particle passes and the symbol "n" when blocked. Based on the expected values, each quantity is then expressed as: $E(\vec{a}, \vec{b}) = Ppp(\vec{a}, \vec{b}) + Pnn(\vec{a}, \vec{b}) - Ppn(\vec{a}, \vec{b}) - Pnp(\vec{a}, \vec{b})$ and, similarly, the other 3 quantities. Before moving further, we may ask ourselves this question: is this description given by normalized density function

$\int \rho(\lambda)\,d\lambda = 1$, justified for the entire population (polarization states over 3600) or any individual instances (if it exists)? Mermin [26] provided some interesting perspectives on hidden variables and the Bell theorem. Batle and Casas [27] computed the density matrix for particles of spin- 1/2 transverse magnetic field and the Bell inequality. They analyzed the entanglement and nonlocality through the maximal violation of a Bell inequality. Scully and Zubairy [28] presented a review on the EPR arguments concerning the incompleteness of quantum mechanics and Bell inequality violation by quantum mechanics. They claimed that a nonlocal hidden variables theory can be developed which is in agreement with quantum theory.

One very important finding in this study is that the Bell inequalities (Equation 1) on normalized density function is true only for photon polarization at individual instances but it does not

hold true over the entire population in a given system. Two independent measurements using hidden variables are not entirely incompatible with the quantum-mechanical predictions. This will be very clear in the next section when we analyze the hidden variables predictions over the entire 360^0 hemisphere and for individual instances. A preprint of this research paper was posted [29].

Hidden Variables

On Photon Polarization, suppose we represent all photon polarization states (population) over the entire hemisphere by N-unit vectors (λ_k, k=1.. N) and interactions by N-orientations for each polarizer

($\vec{a}_i$, i=1.. N, $\vec{b}$, j=1.. N) relative to the photon polarization axis (Figure 1). The expected value of two independent measurements can then be expressed as:

$$P(\vec{a},b) = | A(\vec{a},\lambda)B(b,\lambda)\rho(\lambda)d\lambda | = \sum_{k=1}^{N} A|\vec{a}_i,\lambda_k|B|b_j,\lambda_k| \tag{2}$$

Computationally, we do not have any difficulties in predicting this probability field, $P(\vec{a}_i, \vec{b}_j)$ for any given photon pairs whence ($(\vec{a}_i, \lambda)$) and ($(\vec{b}, \lambda)$) are the corresponding angles between the polarizer orientation and photon state vector (λ_k). For such a setting, the expected values are $A(\vec{a}_i, \lambda) = \cos(\vec{a}_i, \lambda)$; $B(\vec{b}, \lambda) = \cos(\vec{b}, \lambda)$ for photons that passed through filters A and B (similarly, by the sine function when the photon is blocked). Bell viewed this picture from an observation perspective and concluded, "...the statistical features of quantum mechanics arise because the value of this variable is unknown in individual instances."

For an ensemble of all polarization states (over 3600 hemisphere), the four quantities in Bell test (Figure 1) can be defined as:

$$P_{pp}^{\lambda}|\vec{a}_i,b_j| = \sum_{k=1}^{N} \cos|\vec{a}_i,\lambda_k|\cos(\vec{b}_j,\vec{\lambda}_k) \tag{3a}$$

$$P_{nn}^{\lambda}|\vec{a}_i,b_j| = \sum_{k=1}^{N} \sin|\vec{a}_i,\lambda_k|\sin|b_j,\lambda_k| \tag{3b}$$

$$P_{pn}^{\lambda}|\vec{a}_i,b_j| = \sum_{k=1}^{N} \cos|\vec{a}_i,\lambda_k|\sin|b_j,\lambda_k| \tag{3c}$$

$$P_{np}^{\lambda}|\vec{a}_i,b_j| = \sum_{k=1}^{N} \sin|\vec{a}_i,\lambda_k|\cos|b_j,\lambda_k| \tag{3d}$$

The first two equations (3a) and (3b) describe when both photons pass or are blocked, respectively. The last two equations (3c) and (3d) describe when one passes by filter A and the other is blocked by filter B, respectively. Sign "bar" indicates an average (an ensemble of all polarization states).

A systematic prediction on correlated photon pairs will show next that these four equations in fact fully describe four quantities in Bell test and have a complete agreement with the experiments. This finding gives us great satisfaction as perceived naturalness of each element in the Bell test (photon-filter interaction did not violate the special theory of relativity). The hidden

variables predictions were presented step by step in Appendix A.

Quantum Mechanics

On wavefunction superposition, quantum mechanics describes two distance measurements in a unified system. The expected values in such a system can be defined as:

$$P_{pp}^{QM}(\vec{a}_i,\vec{b}_j) = \frac{1}{2}\cos^2(\vec{a}_i,\vec{b}_j) \tag{4a}$$

$$P_{nn}^{QM}(\vec{a}_i,\vec{b}_j) = \frac{1}{2}\sin^2(\vec{a}_i,\vec{b}_j) \tag{4b}$$

The equations 4a and 4b describe when both photons are passed or blocked, respectively. As we can see, quantum mechanics is unable to quantify four measurement quantities in the Bell test. Our objection here is that the quantum mechanical description given by wavefunction is incomplete and it is incapable of providing a complete picture of a physical system (i.e. no intrinsic character). Einstein was not able to accept such a description given by the wavefunction (but unable to provide a concrete theoretical and experimental details).

Bell Test Prediction

This section presented the local hidden variables predictions for each quantity in the Bell test (equations 3a,3b,3c,3d) and compared them with quantum mechanics (equations 4a,4b). The results showed that the hidden variables predictions are not entirely incompatible with quantum mechanics (which is a contradiction with Bell's argument).

Hidden Variables Prediction

There were three main steps in the hidden variables predictions on correlated photons (more details were given in Appendix A): (I) probabilities for photon interaction, $P^{\lambda k}(\vec{a}, \vec{b})$, $P^{\lambda k}(\vec{a}, \vec{b})$, $P^{\lambda k}(\vec{a}, \vec{b})$, $P^{\lambda k}(\vec{a}, \vec{b})$ (assuming 11.25^0 equally spaced intervals (i=1.. 32, j =1.. 32, k=1.. 32) over 360^0 hemisphere).
(II) expected values ($E^{\lambda k}(\vec{a},\vec{b}) = P^{\lambda k}(\vec{a}, \vec{b}) + P^{\lambda k}(\vec{a}, \vec{b}) - P^{\lambda k}(\vec{a}, \vec{b}) - P^{\lambda k}(\vec{a}, \vec{b})$) and Bell inequalities ($S^{\lambda k} = E^{\lambda k}(\vec{a}, \vec{b}) - E^{\lambda k}(\vec{a}, \vec{b}') + E^{\lambda k}(\vec{a}', \vec{b}) + E^{\lambda k}(\vec{a}', \vec{b}')$)) for each polarization states.
(III) average expected values ($\bar{E}(\vec{a}_i,\vec{b}) = \bar{P}_{pp}(\vec{a}_i, \vec{b}) + \bar{P}_{nn}(\vec{a}_i, b) - \bar{P}_{pn}(\vec{a}_i,b) - \bar{P}_{np}(\vec{a}_i, b)$) and Bell inequalities ($\bar{S} = \bar{E}(\vec{a}, b) - \bar{E}(\vec{a}, b') + \bar{E}(\vec{a}',b) + \bar{E}(\vec{a}', b')$)) for the entire population (as many Bell tests conducted). Equations 3a to 3b in Section 3.2 were used to calculate four normalized probabilities.

For clarity (such as index (i,j,k), photon interaction angle (ai,λk)), the following is a sample calculation for photons within a visible spectrum: cos(a3,λ2)cos(b3,λ2) = cos(2πcos(acos(a3-λ2)/180)*cos(2πcos(acos(b3- λ2)/180) = 0.854 when λ2= 11.25^0(correlated photons), a3=22.50, b3=22.50. It should be noted that, in quantum mechanics, the Bell inequalities for a unified system were calculated using the probability Equations (4a, 4b).
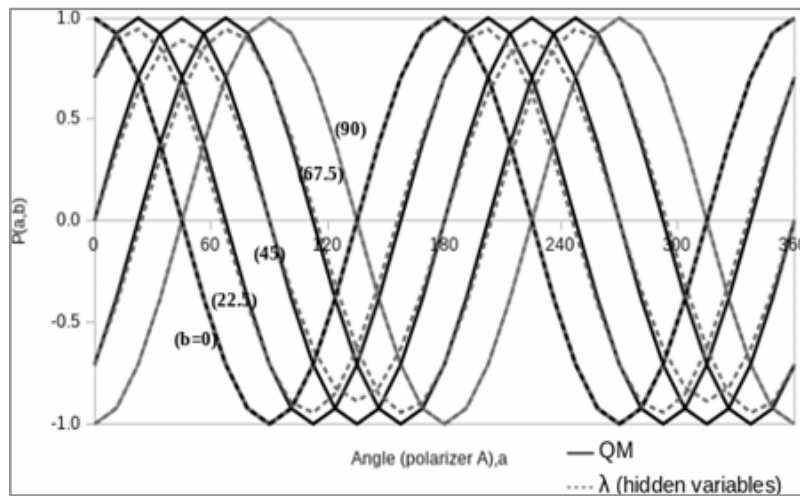
Hidden Variables vs Quantum Mechanics

In quantum mechanics, two equations (4a, 4b) are describing the Bell test (a unified system). In hidden variables, four equations

(3a to 3d) are describing the Bell test (each quantity). Let us compare the hidden variables prediction with the quantum mechanics when both photons passed (equation 3a vs equation 4a). These two predictions, $\overline{P^\lambda}(\vec{a},\vec{b}) \equiv \overline{P^{QM}}(\vec{a},\vec{b})$, are nearly identical. Figure 2 showed a
sample comparison for five selected orientations of polarizer B $B(\vec{b},\vec{b},\vec{b},\vec{b},\vec{b}))$ and all 32 orientations of polarizer A.

When photons passed one filter and block by another, the results showed very similar agreement with quantum mechanics. Interestingly, there are some well defined disagreements (small to large periodically). This disagreement is not the same as Bell's incompatibility argument between local causality vs quantum mechanics.

Figure 2: Probabilities when photons passed both filters - Local hidden variables (equation 3a) vs quantum mechanics (equation 4a).

This overall agreement (local hidden variables vs quantum mechanics) is a contradiction with Bell's argument that the hidden variables prediction is incompatible with quantum mechanics. Next, we will get concrete proof that the local hidden variables (photon polarization) prediction given by the equations 3a-3d is correct (physical) and the quantum mechanics prediction given by the equations 4a,4b is a system approximation (not physical). The conclusion is that the Bell inequalities is not an incompatibility criterion for local hidden variables vs quantum mechanics rather it is a criterion for an individual nature vs population dynamics.
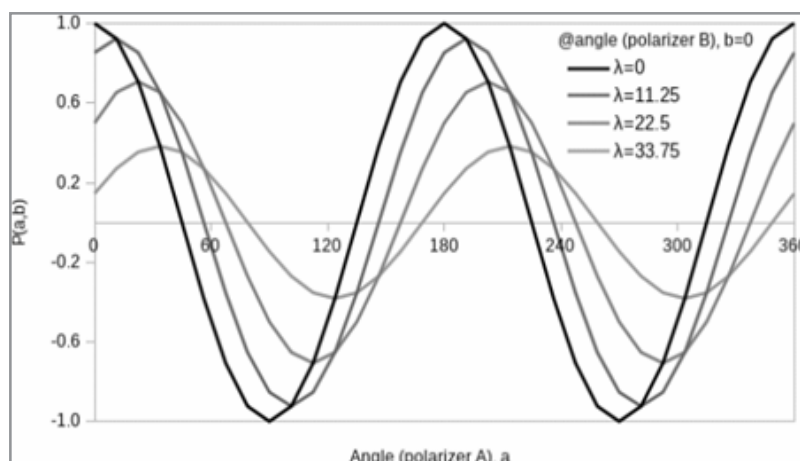
Bell Inequalities Prediction
In the scientific communities, it is well known that the Bell inequalities on local hidden variables will always be bounded by this limit ($-2 \leq S \leq 2$). And the quantum mechanics can not be replaced by a theory that uses hidden variables. The results here will show that this claim no longer holds. The Bell inequalities,

$-2 \leq S \leq 2$, is true only for photon polarization at individual instances but it does not hold over the entire population (a given system). In hidden variables, each photon polarization is predictable (measurable) physical property. An outcome of a system (or population dynamics) is defined by an ensemble of this property over the entire spectrum.

Individual Polarization
On Individual Instances, let us analyze a few details (such as probabilities, the Bell inequalities) for photons at individual instances (λ_k, $k = 1..32$). Figure 3 showed a sample of the expected values for several individual instances ($\lambda_1 = 0$, $\lambda_2 = 11.25$, $\lambda_3 = 22.5$, $\lambda_4 = 33.75$) when the polarizer B orientation vector is set to $\vec{b}_1 = 0$ and polarizer A for all orientation vectors ($\vec{a}_i$, $i = 1..32$). Such hidden variables predictions for individual polarization have no realization in quantum mechanics. The prediction for each photon may support Einstein pictures of physical reality.

Figure 3: Hidden variables prediction (equation 3a) for photons passed both filters at four selected polarization states (intrinsic properties, no realization in quantum mechanics).

Table 1 summarized the Bell inequalities for several polarizer settings at observation sites A and B (only the upper and lower values are shown). The hidden variables predictions fully agreed with the Bell inequalities ($-2 \leq S \leq 2$). The inequalities in Table 1 can be experimentally verified if we can somehow set up a test for photon pairs over a specified polarization (such as an ensemble of the state vectors, λ, over a selected interval of $\theta=11.25$). Using the setup in Figure 1, such a test may be possible if we use an optical filter that lets photons of a specified polarization pass through while blocking photons of other polarization (before entering two distance measurements). The criterion in designing any such optical filter should be blocking all photons of unwanted polarization and allowing only photons of specific polarization (any disturbances need to be justified). Until now, the Bell inequalities for individual instances were not known and Bell tests were conducted over the entire 360 0 hemisphere (as given in next section).

Table 1: Bell inequalities for photon polarization at individual instances (shown only the limits for each computational setup) (Table A-5 in appendix A showed all values, $S\lambda k$, for individual polarization).
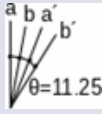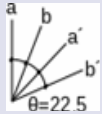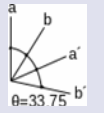
Polarizers setup $\theta=(a,b)=(b,a')=(a',b'))=(a')=(a',b'),b')=(a',b'))$	S limit for each setup (32 S values for 32 polarization states)
$\theta=11.25, 78.25$	$0.541 \leq S \leq 1.848, -1.848 \leq S \leq -0.541$
$\theta=22.5, 67.5$	$S=1.414, S \leq -1.414$
$\theta=33.75, 56.25$	$0.383 \leq S \leq 1.689, -1.689 \leq S \leq -0.383$
$\theta=45, 135$	$-1.0 \leq S \leq 1.0$
$\theta=90, 180$	$-2.0 \leq S \leq 2.0$

Polarization Over 3600 Hemisphere

Over 3600 hemisphere (population), the predictions (Table 2) for several orientations of the polarizers showed a clear violation of the Bell inequalities ($-2 \leq S \leq 2$). Most surprisingly, the hidden variable predictions are in complete agreement with the quantum mechanics and Aspect et al. prediction on lab data. This is in contradiction with an affirmative conclusion in favor of quantum mechanics made by the experimental studies (until now).

Table 2: Bell inequalities (hidden variables, quantum mechanics (QM), lab data) over the entire 360^0 hemisphere (ensemble of all polarization states) (see Table A-7 in appendix A for more details).

Test			
λ (hidden variables)§ QM Aspect et al. Exp.	2.328 2.389 -	2.794 2.828 2.697±0.015	2.047 2.072 -

§hidden variables prediction (photon polarization) in this study (previously not known).

Table 3 showed the Bell test analysis for two independent Bell experiments (relatively large scale setup) and the corresponding predictions. The hidden variables predictions here are consistent with the experimental quantities and have a complete agreement. In contrast, the following measured quantities in Bell test have no realization in quantum mechanics (a unified system) description: E(a,b), E(a,b ′), E(a′,b) and E(a′,b′). This clearly demonstrates that the local hidden variables (photon polarization states) prediction is correct (real) and the quantum mechanics prediction on a unified system is an approximation (not physical).

Table 3: Bell test analysis (hidden variables, quantum mechanics (QM), lab data) over the entire 3600 hemisphere (ensemble of all polarization states).

	E(a,b)	E(a,b)	E(a′,b)	E(a′,b′)	S
λ (hidden variables)§ QM Cosmic Bell Test Laser Entangled Photons	0.686 - 0.670 0.559	-0.728 - -0.739 -0.591	0.669 - 0.637 0.560	0.711 - 0.628 0.820	2.794 2.828 2.674 2.530

§hidden variables prediction (photon polarization) in this study (previously not known).

The Cosmic Bell Test was conducted by a team of Anton Zeilinger. This is a large version of Clauser and Freedman's Bell test experiment. In this Cosmic Bell test, the source of the entangled particles was about a third of a mile from each of the detectors. The team sent perfectly timed pairs of photons through the air to each detector. At the same time, the telescopes collect light from two extremely far- off, extremely bright galaxies called quasars. Random variations in this light controlled which filters were used to measure the photon pairs. In such controlling, it was incredibly unlikely that anything could be influencing the random nature of the filters. The Laser Entangled Photons Bell test was presented in a documentary on quantum physics, by Jim Al-Khalili (very informative for the scientific communities and for the general public). This is a modern version of an experiment first carried out by John Clauser and then Alain Aspect. In this experiment, a crystal converts laser light into pairs of entangled light quanta, photons, making two very precise beams. These correlated photons then passed through the detectors.

Summary Results

Table 1 presented the Bell inequalities (local hidden variables) for photon polarization at individual instances. The predictions are bounded by the Bell inequalities, $-2 \leq S \leq 2$. This is because the following two systems are identical: the prediction system (photon state vector at an individual instance) and the Bell system (ensemble of the states vectors by normalized density function). Table 2 presented the Bell inequalities for the entire 3600 hemisphere. The local hidden variables predictions have a complete agreement with the experiments and quantum mechanics. The results (hidden variables, quantum mechanics, lab data) showed a clear violation of the Bell inequalities, $-2 \leq S \leq 2$, The results reject the following conclusion made by the earlier studies: quantum mechanics can not be replaced by a theory that uses hidden variables. Table 3 showed each quantity of the Bell inequalities and a complete agreement with the experimental predictions. This clearly demonstrated that the photon-filter interaction was a local cause (no violation of special theory of relativity) (as Einstein expected).

The Bell inequalities, $-2 \leq S \leq 2$, was derived based on the normalized density function for the correlated photon pairs or spin half particles. As a result, the Bell inequality is unable to distinguish any individual photon state (or any group of polarization states) from its population in a given experiment. The normalized density assumption may also mislead the violation Bell inequalities in classical optical setup or in large scale molecular dynamics (some perspectives on "individual instances vs population" in particles and biological systems are given in appendix B). A case study in appendix B from our everyday experience analogous to the correlated photon pairs experiment as well as double slits experiment. This demonstrated a correlation "individual instances vs population" in particles and biological systems that may help to better understand the Bell inequalities on local causality.

Conclusion

This study revealed that the Bell inequalities, $-2 \leq S \leq 2$, on normalized density function is correct for photon polarization at individual instances but it does not hold over the entire population in a given system. The inequalities violation occurred in both quantum mechanics (unified system) and local hidden variables (photon polarization). This is because a unified system in quantum mechanics and photons population in hidden variables are nearly identical. For polarization at individual instances, the hidden variables prediction was found to be bounded by the Bell inequalities. This study believes that Bell test on photon polarization at individual instances may be possible if we can overcome measurement disturbances. Most importantly, this study arrived at the following conclusion with great satisfaction (as perceived naturalness of each element in the Bell test): the local hidden variables (photon polarization) prediction is correct (real) and the quantum mechanics description given by wavefunction in a unified system is an approximation (not physical). The interaction of photons at a distance was a local cause (no violation of the special theory of relativity).

Competing Interest

The author declares no known competing interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A : Local Hidden Variables Prediction
Photon Filter Interaction

Equations A-1 and A-2 describe when both photons pass (subscript "pp") or are blocked (subscript "nn"), respectively. Equations (A-3) and (A-4) describe when one passes by filter A and the other is blocked by filter B (subscript "pn" or "np"), respectively. The indexes i, j represent the filter orientations, and k for photon polarization state.

$$P_{pp}^{\lambda k}(\vec{a}_i,\vec{b}_j)=\cos(\vec{a}_i,\vec{\lambda}_k)\cos(\vec{b}_j,\vec{\lambda}_k) \qquad \text{(A-1)}$$

$$P_{nn}^{\lambda k}(\vec{a}_i,\vec{b}_j)=\sin(\vec{a}_i,\vec{\lambda}_k)\sin(\vec{b}_j,\vec{\lambda}_k) \qquad \text{(A-2)}$$

$$P_{pn}^{\lambda k}(\vec{a}_i,\vec{b}_j)=\cos(\vec{a}_i,\vec{\lambda}_k)\sin(\vec{b}_j,\vec{\lambda}_k) \qquad \text{(A-3)}$$

$$P_{np}^{\lambda k}(\vec{a}_i,\vec{b}_j)=\sin(\vec{a}_i,\vec{\lambda}_k)\cos(\vec{b}_j,\vec{\lambda}_k) \qquad \text{(A-4)}$$

Bell Inequalities Prediction
Individual Polarization

For experimental setup (Figure 1 in main body), the bell inequalities was given as:

$$S^{\lambda k}=E^{\lambda k}(\vec{a}_i,\vec{b}_j)-E^{\lambda k}(\vec{a}_i,\vec{b'}_j)+E^{\lambda k}(\vec{a'}_i,\vec{b}_j)+E^{\lambda k}(\vec{a'}_i,\vec{b'}_j) \qquad \text{(A-5a)}$$

where the expected value was defined as:

$$E^{\lambda k}(\vec{a}_i,\vec{b}_j)=P_{pp}^{\lambda k}(\vec{a}_i,\vec{b}_j)+P_{nn}^{\lambda k}(\vec{a}_i,\vec{b}_j)-P_{pn}^{\lambda k}(\vec{a}_i,\vec{b}_j)-P_{np}^{\lambda k}(\vec{a}_i,\vec{b}_j)|$$

(A-5b)

Over Population

The bell inequalities for the entire population (using all polarization states over 3600 hemisphere) was defined as:

$$\overline{S}=\overline{E(\vec{a},\vec{b})}-\overline{E(\vec{a},\vec{b})}+\overline{E(\vec{a'},\vec{b})}+\overline{E(\vec{a'},\vec{b'})} \qquad \text{(A-6a)}$$

where the expected value is an average over all polarization states as:

$$E(\vec{a}_i,\vec{b})=\overline{P_{pp}}(\vec{a}_i,\vec{b})+\overline{P_{nn}}(\vec{a}_i,\vec{b})-\overline{P_{pn}}(\vec{a}_i,\vec{b})-\overline{P}_{np}(\vec{a}_i,\vec{b}) \qquad \text{(A-6b)}$$

applied for all polarization states (k) and filter orientations (i, j) over 360^0 (evenly spaced).

Photon Interaction Prediction
Tables A-1 to A-4 showed the photon interaction predictions using the equation A-1 to A-4, respectively. This approach was

Table A-1: Prediction, $P^{\lambda k}(\vec{a},\vec{b})$, for polarization state at λ2=11.250.

ai/bj	b1 (0)	b2 (11.25)	b3 (22.5)	b4 (33.75)	b5 (45)	b6 (56.25)	b7 (67.5)	b8 (78.75)	b9 (90)	. b32 (348.75)
λ2=11.25										
a1 (0)	0.854	0.924	0.854	0.653	0.354	0.0	-0.354	-0.653	-0.854	0.653
a2 (11.25)	0.924	1	0.924	0.707	0.383	0.0	-0.383	-0.707	-0.924	0.707
a3 (22.5)	0.854	0.924	0.854	0.653	0.354	0.0	-0.354	-0.653	-0.854	0.653
a4 (33.75)	0.653	0.707	0.653	0.5	0.271	0.0	-0.271	-0.5	-0.653	0.5
a5 (45)	0.354	0.383	0.354	0.271	0.146	0.0	-0.146	-0.271	-0.354	0.271

a6 (56.25)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
a7 (67.5)	-0.354	-0.383	-0.354	-0.271	-0.146	0.0	0.146	0.271	0.354	-0.271
a8 (78.75)	-0.653	-0.707	-0.653	-0.5	-0.271	0.0	0.271	0.5	0.653	-0.5
a9 (90)	-0.854	-0.924	-0.854	-0.653	-0.354	0.0	0.354	0.653	0.854	-0.653
.										
a32 (348.75)	0.653	0.707	0.653	0.5	0.271	0.0	-0.271	-0.5	-0.653	0.5

For clarity, this is a sample calculation: cos(a3,λ2)cos(b3,λ2) = cos(2πcos(acos(a3-λ2)/180)*cos(2πcos(acos(b3-λ2)/180) = 0.854 when λ1= 11.250 (correlated photons), a3=22.5, b3=22.5.

Table A-2: Prediction, $P^{\lambda k}(\vec{a},\vec{b})_n$ for polarization state at λ2=11.250.

ai/bj	b1 (0)	b2 (11.25)	b3 (22.5)	b4 (33.75)	b5 (45)	b6 (56.25)	b7 (67.5)	b8 (78.75)	b9 (90)	b32 (348.75).
λ1= 0										
λ2= 11.25										
a1 (0)	0.146	0.0	-0.146	-0.271	-0.354	-0.383	-0.354	-0.271	-0.146	0.271
a2 (11.25)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
a3 (22.5)	-0.146	0.0	0.146	0.271	0.354	0.383	0.354	0.271	0.146	-0.271
a4 (33.75)	-0.271	0.0	0.271	0.5	0.653	0.707	0.653	0.5	0.271	-0.5
a5 (45)	-0.354	0.0	0.354	0.653	0.854	0.924	0.854	0.653	0.354	-0.653
a6 (56.25)	-0.383	0.0	0.383	0.707	0.924	1	0.924	0.707	0.383	-0.707
a7 (67.5)	-0.354	0.0	0.354	0.653	0.854	0.924	0.854	0.653	0.354	-0.653
a8 (78.75)	-0.271	0.0	0.271	0.5	0.653	0.707	0.653	0.5	0.271	-0.5
a9 (90)										
.	-0.146	0.0	0.146	0.271	0.354	0.383	0.354	0.271	0.146	-0.271
a32 (348.75)	0.271	0.0	-0.271	-0.5	-0.653	-0.707	-0.653	-0.5	-0.271	0.5

Table A-3: Predic-tion $P^{\lambda k}(\vec{a},\vec{b})$ for polarization state at λ2=11.250.

ai/bj	b1 (0)	b2 (11.25)	b3 (22.5)	b4 (33.75)	b5 (45)	b6 (56.25)	b7 (67.5)	b8 (78.75)	b9 (90)	b32 (348.75)
λ2=11.25										
a1 (0)	-0.354	0.0	0.354	0.653	0.854	0.924	0.854	0.653	0.354	-0.653
a2 (11.25)	-0.383	0.0	0.383	0.707	0.924	1	0.924	0.707	0.383	-0.707
a3 (22.5)	-0.354	0.0	0.354	0.653	0.854	0.924	0.854	0.653	0.354	-0.653
a4 (33.75)	-0.271	0.0	0.271	0.5	0.653	0.707	0.653	0.5	0.271	-0.5
a5 (45)	-0.146	0.0	0.146	0.271	0.354	0.383	0.354	0.271	0.146	-0.271
a6 (56.25)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

a7 (67.5)	0.146	0.0	-0.146	-0.271	-0.354	-0.383	-0.354	-0.271	-0.146	0.271
a8 (78.75)	0.271	0.0	-0.271	-0.5	-0.653	-0.707	-0.653	-0.5	-0.271	0.5
a9 (90)										
.	0.354	0.0	-0.354	-0.653	-0.854	-0.924	-0.854	-0.653	-0.354	0.653
a32 (348.75)	-0.271	0.0	0.271	0.5	0.653	0.707	0.653	0.5	0.271	-0.5

Table A-4: Predic-tion , $P^{\lambda_k}\left(\vec{a},\vec{b}\right)$, for polari-zation stat e at $\lambda_2 = 11.250$.

ai/bj	b1 (0)	b2 (11.25)	b3 (22.5)	b4 (33.75)	b5 (45)	b6 (56.25)	b7 (67.5)	b8 (78.75)	b9 (90)	b32 (348.75)
λ2=11.25										
a1 (0)	-0.354	-0.383	-0.354	-0.271	-0.146	0.0	0.146	0.271	0.354	-0.271
a2 (11.25)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
a3 (22.5)	0.354	0.383	0.354	0.271	0.146	0.0	-0.146	-0.271	-0.354	0.271
a4 (33.75)	0.653	0.707	0.653	0.5	0.271	0.0	-0.271	-0.5	-0.653	0.5
a5 (45)	0.854	0.924	0.854	0.653	0.354	0.0	-0.354	-0.653	-0.854	0.653
a6 (56.25)	0.924	1	0.924	0.707	0.383	0.0	-0.383	-0.707	-0.924	0.707
a7 (67.5)	0.854	0.924	0.854	0.653	0.354	0.0	-0.354	-0.653	-0.854	0.653
a8 (78.75)	0.653	0.707	0.653	0.5	0.271	0.0	-0.271	-0.5	-0.653	0.5
a9 (90)										
.	0.354	0.383	0.354	0.271	0.146	0.0	-0.146	-0.271	-0.354	0.271
a32 (348.75)	-0.653	-0.707	-0.653	-0.5	-0.271	0.0	0.271	0.5	0.653	-0.5

Bell Inequalities (Individual Polarization)

For all settings (i,j,k), the expected values were predicted using Equation A-5b. The Bell inequalities were then calculated for several selected cases. Table A-5 showed the expected values for λ2=11.250 (as a demonstration). Table A-6 summarized the Bell inequalities for individual polarization (discussed in main body Table 3).

Table A-5: Prediction, $E^{\lambda_k}\left(\vec{a},\vec{b}\right)$, for polarization state at $\lambda_2 = 11.250$.

ai/bj	b1 (0)	b2 (11.25)	b3 (22.5)	b4 (33.75)	b5 (45)	b6 (56.25)	b7 (67.5)	b8 (78.75)	b9 (90)	. b32 (348.75)
λ2=11.25										
a1 (0)	0.854	0.653	0.354	0.0	-0.354	-0.653	-0.854	-0.924	-0.854	0.924
a2 (11.25)	0.653	0.5	0.271	0.0	-0.271	-0.5	-0.653	-0.707	-0.653	0.707
a3 (22.5)	0.354	0.271	0.146	0.0	-0.146	-0.271	-0.354	-0.383	-0.354	0.383
a4 (33.75)	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
a5 (45)	-0.354	-0.271	-0.146	0.0	0.146	0.271	0.354	0.383	0.354	-0.383
a6 (56.25)	-0.653	-0.5	-0.271	0.0	0.271	0.5	0.653	0.707	0.653	-0.707
a7 (67.5)	-0.854	-0.653	-0.354	0.0	0.354	0.653	0.854	0.924	0.854	-0.924
a8 (78.75)	-0.924	-0.707	-0.383	0.0	0.383	0.707	0.924	1	0.924	-1
a9 (90)	-0.854	-0.653	-0.354	0.0	0.354	0.653	0.854	0.924	0.854	-0.924
.										

a32 (348.75)	0.924	0.707	0.383	0.0	-0.383	-0.707	-0.924	-1	-0.924	1

Table A-5: The Bell Inequalities, $S\lambda k$, for individual polarization states at several different setting.

	a1b2a3b4 (11.25)	a1b3a5b7 (22.5)	a1b4a7b10 (33.75)	a1b5a9b13 (45)	a1b6a11b16 (56.25)	a1b7a13b19 (67.5)	a1b8a15b22 (78.75)	a1b9a17b25 (90)	a1b10a19b28 (101.25)	a1b11a21b31 (112.5)
5)	0.5412	1.4142	0.7654	-1.0000	-1.3066	-1.4142	-1.8478	-1.0000	-0.5412	-1.4142
λ2=11.25	0.9239	1.4142	1.3066	-0.7071	-1.6892	-1.4142	-1.8478	-1.7071	-0.9239	-1.4142
λ3=22.5	1.4651	1.4142	1.6892	0.0000	-1.6892	-1.4142	-1.4651	-2.0000	-1.4651	-1.4142
λ4=33.75	1.8478	1.4142	1.6892	0.7071	-1.3066	-1.4142	-0.9239	-1.7071	-1.8478	-1.4142
λ5=45	1.8478	1.4142	1.3066	1.0000	-0.7654	-1.4142	-0.5412	-1.0000	-1.8478	-1.4142
λ6=56.25	1.4651	1.4142	0.7654	0.7071	-0.3827	-1.4142	-0.5412	-0.2929	-1.4651	-1.4142
λ7=67.5	0.9239	1.4142	0.3827	0.0000	-0.3827	-1.4142	-0.9239	0.0000	-0.9239	-1.4142
λ8=78.75	0.5412	1.4142	0.3827	-0.7071	-0.7654	-1.4142	-1.4651	-0.2929	-0.5412	-1.4142
λ9=90	0.5412	1.4142	0.7654	-1.0000	-1.3066	-1.4142	-1.8478	-1.0000	-0.5412	-1.4142
λ10=101.25	0.9239	1.4142	1.3066	-0.7071	-1.6892	-1.4142	-1.8478	-1.7071	-0.9239	-1.4142
λ11=112.5	1.4651	1.4142	1.6892	0.0000	-1.6892	-1.4142	-1.4651	-2.0000	-1.4651	-1.4142
λ12=123.75	1.8478	1.4142	1.6892	0.7071	-1.3066	-1.4142	-0.9239	-1.7071	-1.8478	-1.4142
λ13=135	1.8478	1.4142	1.3066	1.0000	-0.7654	-1.4142	-0.5412	-1.0000	-1.8478	-1.4142
λ14=146.25	1.4650	1.4142	0.7655	0.7069	-0.3829	-1.4142	-0.5414	-0.2931	-1.4650	-1.4142
λ15=157.5	0.9239	1.4142	0.3827	0.0000	-0.3827	-1.4142	-0.9239	0.0000	-0.9239	-1.4142
λ16=168.75	0.5412	1.4142	0.3827	-0.7071	-0.7654	-1.4142	-1.4651	-0.2929	-0.5412	-1.4142
λ17=180	0.5412	1.4142	0.7654	-1.0000	-1.3066	-1.4142	-1.8478	-1.0000	-0.5412	-1.4142
λ18=191.25	0.9239	1.4142	1.3066	-0.7071	-1.6892	-1.4142	-1.8478	-1.7071	-0.9239	-1.4142
λ19=202.5	1.4651	1.4142	1.6892	0.0024	-1.6892	-1.4142	-1.4651	-2.0000	-1.4651	-1.4142
λ20=213.75	1.8478	1.4142	1.6892	0.7071	-1.3066	-1.4142	-0.9239	-1.7071	-1.8478	-1.4142
λ21=225	1.8478	1.4142	1.3066	1.0000	-0.7654	-1.4142	-0.5412	-1.0000	-1.8478	-1.4142
λ22=236.25	1.4651	1.4142	0.7654	0.7071	-0.3827	-1.4142	-0.5412	-0.2929	-1.4651	-1.4142
λ23=247.5	0.9239	1.4142	0.3827	0.0000	-0.3827	-1.4142	-0.9239	0.0000	-0.9239	-1.4142

λ24=258.75	0.5412	1.4142	0.3827	-0.7071	-0.7654	-1.4142	-1.4651	-0.2929	-0.5412	-1.4142
λ25=270	0.5412	1.4142	0.7654	-1.0000	-1.3066	-1.4142	-1.8478	-1.0000	-0.5412	-1.4142
λ26=281.25	0.9239	1.4142	1.3066	-0.7071	-1.6892	-1.4142	-1.8478	-1.7071	-0.9239	-1.4142
λ27=292.5	1.4651	1.4142	1.6892	0.0000	-1.6892	-1.4142	-1.4651	-2.0000	-1.4651	-1.4142
λ28=303.75	1.8478	1.4142	1.6892	0.7071	-1.3066	-1.4142	-0.9239	-1.7071	-1.8478	-1.4142
λ29=315	1.8478	1.4142	1.3066	1.0000	-0.7654	-1.4142	-0.5412	-1.0000	-1.8478	-1.4142
λ30=326.25	1.4651	1.4142	0.7654	0.7071	-0.3827	-1.4142	-0.5412	-0.2929	-1.4651	-1.4142
λ31=337.5	0.9239	1.4142	0.3827	0.0000	-0.3827	-1.4142	-0.9239	0.0000	-0.9239	-1.4142
λ32=348.75	0.5412	1.4142	0.3827	-0.7071	-0.7654	-1.4142	-1.4651	-0.2929	-0.5412	-1.4142
	0.541≤S≤1.848	S=1.414	0.383≤S≤1.689	-1.0≤S≤1.0	-1.689≤S≤-0.383	S=-1.414	-1.848≤S≤-0.541	-2.0≤S≤0.0	1.848≤S≤0.541	S=-1.414

Bell Inequalities (Population)

Table A-6 showed the mean expected values (Equation A-6b) over all polarization states (population). Table A-7 showed the predicted Bell inequalities (Equation A-6a) for the system (over 3600 hemisphere.

Table A-6: The mean expected values, $\overline{E}(\vec{a}_i, \vec{b})$, over all polarization states.

ai/bj	b1 (0)	b2 (11.25)	b3 (22.5)	b4 (33.75)	b5 (45)	b6 (56.25)	b7 (67.5)	b8 (78.75)	b9 (90)	. b32 (348.75)
a1 (0)	1	0.913	0.686	0.356	-0.029	-0.41	-0.728	-0.935	-1	0.935
a2 (11.25)	0.913	0.976	0.89	0.669	0.347	-0.029	-0.401	-0.711	-0.913	0.711
a3 (22.5)	0.686	0.89	0.958	0.881	0.67	0.356	-0.012	-0.378	-0.686	0.378
a4 (33.75)	0.356	0.669	0.881	0.958	0.891	0.686	0.378	0.012	-0.356	-0.012
a5 (45)	-0.029	0.346	0.669	0.89	0.976	0.913	0.711	0.401	0.029	-0.401
a6 (56.25)	-0.41	-0.029	0.356	0.686	0.913	1	0.935	0.728	0.41	-0.728
a7 (67.5)	-0.728	-0.401	-0.012	0.378	0.711	0.935	1	0.944	0.728	-0.944
a8 (78.75)	-0.935	-0.711	-0.378	0.012	0.401	0.728	0.944	1	0.935	-1
a9 (90)										
.	-1	-0.913	-0.686	-0.356	0.03	0.41	0.728	0.935	1	-0.935
a32 (348.75)	0.935	0.711	0.378	-0.012	-0.401	-0.728	-0.944	-1	-0.935	1

Table A-7: The Bell Inequalities, ̄S, for the system (population) at several test cases.

Test	(a,b,a',b')	S	Test	(a,b,a',b')	S
1	(a1,b2,a3,b4) = 11.25	2.3281	6	(a1,b7,a13,b19) = 67.5	-2.7941
§2	(a1,b3,a5,b7) = 22.5	2.7941	7	(a1,b8,a15,b22) = 78.75	-2.4049
3	(a1,b4,a7,b10) = 33.75	2.0467	8	(a1,b9,a17,b25) = 90	-2.0000
4	(a1,b5,a9,b13) = 45	-0.0587	9	(a1,b10,a19,b28) = 101.25	-2.3281
5	(a1,b6,a11,b16) = 56.25	-2.0786	10	(a1,b11,a21,b31) = 112.5	-2.7941

Several lab tests reported the Bell inequalities for this setting (a complete agreement with this hidden variables prediction).

Appendix B: Case Study on Local Causality

This supporting materials included "a case study" from our everyday experience analogous to the correlated photon pairs experiment as well as double slits experiment. This showed us a correlation "individual instances vs population" in particles and biological systems that may help to better understand the Bell inequalities on local causality.
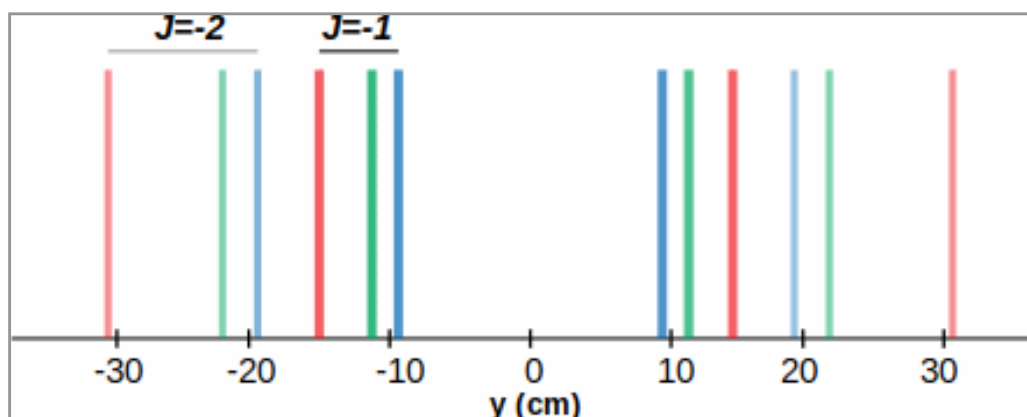
Particles

Before presenting an interesting case study, let us review this very basic optical diffraction result first. Suppose light passes through a double slit of d=0.01mm or through diffraction gratings with 1000 lines per centimeter. Photons are being detected on the screen at x=2.0 m from the slit. The detection results and prediction steps are given in Table B-1. For this illustration, only 3 wavelengths in the visible spectrum are shown. Fig. B-1 shows the photon detection spectrum. As a photon passes, it spreads as a function of wavelength (λ) and slit size (d).

In a real experiment, things become a little interesting, suppose we are sending one photon at a time and recording the detection. The results will look random initially due to the limited number of data points, after which interference patterns will appear. Such interference patterns can easily be described by applying wavefunction superposition. Let us conclude this simple test with the following remark: what happens if we apply the computation steps in Table B-1 by incorporating the initial details of each photons (such as polarization, momentum)? Can we obtain the same prediction as given by wavefunction superposition? The case study given next will shed some light on this question. Our main objective here is to clarify any uncertainties between local causalities and nonlocality.

Table B-1. Basic light diffraction experiment.

Experiment Setup	J_i	λ_i (nm)	$\theta_i = \sin^{-1}(J \lambda_i / d_i)$ (degree)	$y_i = x \tan(\theta_i)$ (cm)
d = 0.01mm, x = 2m, J = 1,-1,2,-2...	1	485 (blue) 565 (green) 750 (red)	2.78 3.24 4.30	9.71 11.32 15.04
	2	485 565 750	5.57 6.49 8.63	19.49 22.75 30.34

Figure B-1: Visible light spectrum from the diffraction experiment.

Athletes

This test case involves local causality. The objective is to demonstrate how an intrinsic property for a given system appears to an observer as nonlocal. The physical interpretation given by wavefunction superposition for many optical entanglements on nolocality may not be the physical reality.

Let us first define a physical system. A system can be any elementary particle, molecule or biological species where each system has an individual identity. When a person such as an athlete is trained for a given race, the person is conditioned on his or her training program, which ultimately determines the expected outcomes. The variance in the expected outcomes is much narrower for a highly trained professional than for nonprofessional athletes.

With respect to great curiosity, this study analyzed the performance completion times of 26,260 trained athletes from a highly recognized annual event (Boston Marathon 2023). Among these participants, there were 14,911 males and 11,349 females. All the participants were highly trained athletes who were challenging themselves (i.e., aiming for personal best). The statistical analysis of the participants' performance on age and sex revealed some interesting correlations that were very similar to the light diffraction interference patterns.

Table B-2 summarizes the statistical analysis of the finishing times for several age groups. Let us check the first age group (18-39); here, a total of 5,285 men had an average finish time of μ=196.50 minutes with a standard deviation of σ=42.50; 4,884 women had an average finish time of μ=227.91 minutes with a standard deviation of σ=42.76. The distribution was slightly left skewed if we compared the median with the mean value. We plotted the means for all age groups only as a line diagram (Figure B- 2). This result looks very similar to many double-slit interference experiments. Interestingly, the overall time difference between men and women was 25 min, which was close to the 30 min set in the qualifying entry. The first age group was 4.2 times wider than the other age groups were. All the lead athletes were within this age group (18-39). Certainly, there should be a more visible spectrum (similar to Figure B-2) if we further analyze the distribution of this group. A photon experiment on "a delayed choice quantum eraser" by Kim et al. (2000) may be analogous to this case study. This case study is a good analogues illustration of Bell's characterization of photon polarization states (state vector at individual instance vs all states vectors over the entire 360 degree hemisphere).

In conclusion, a physical reality (such as intrinsic properties) may appear visible by repeating a large number of tests if such properties have any weighted interaction with the observation setup. One may think what this case study has anything to do with the hidden variables (i.e., intrinsic properties) we are dealing with in physics. Einstein believes that every physical reality of any given system can be predicted with 100% certainty by a complete theory. An attempt here is to demonstrate that the nonlocality arguments on entanglements observed in many experiments are inconclusive.

Table B-2: Statistical analysis of the completion time of the highly conditioned athletes (data from a recognized annual event).

Age Group	Men				Women			
	N	μ (min)	σ	Median	N	μ (min)	σ	Median
18-39	5285	196.50	42.50	179.60	4884	227.91	42.76	212.38
40-44	2241	203.00	39.15	189.28	1878	230.76	37.57	219.21
45-49	2252	210.26	37.09	198.97	1817	235.87	33.18	227.17
50-54	2052	221.32	40.38	207.39	1207	244.61	35.03	233.92
55-59	1422	228.29	37.27	217.36	815	253.28	35.01	244.13
60-64	1110	239.82	36.79	229.95	527	258.9	31.50	252.77
65-69	549	252.95	37.06	242.58	221	272.10	30.34	267.75
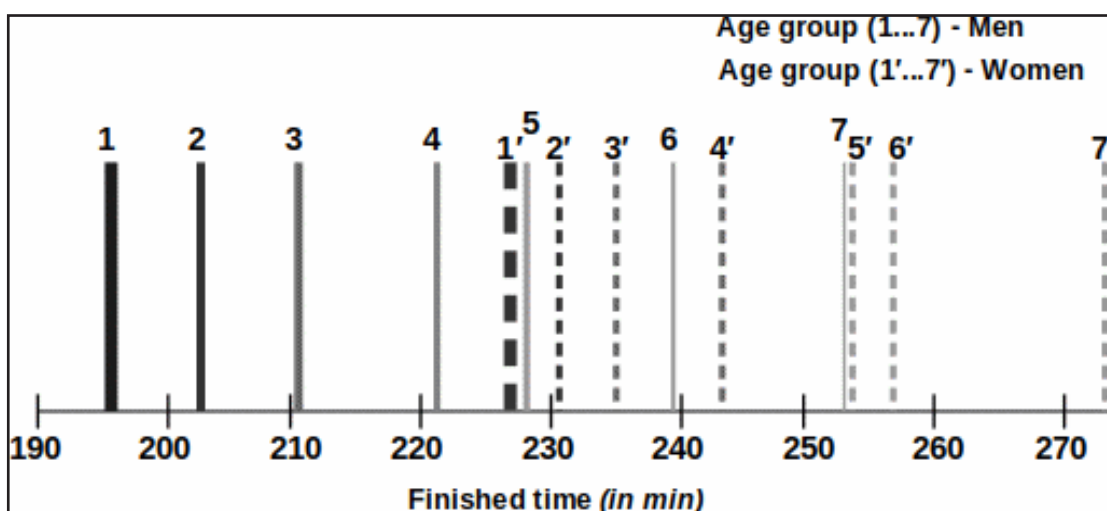
Figure B-2: Mean completion times by age group and sex (highlighting performance status).

This study claimed that often, a physical reality (i.e., intrinsic properties) can be visualized or detectable only by repeating a large number of test experiments. Test examples from single-photon diffraction experiments and the performances of a large number of athletes at an event clearly showed how the intrinsic properties appeared as a visible spectrum in the observations. An overall population response here depends on the relative positions of the particles (particles system) or athletes (sports event) by their intrinsic properties. The Bell inequalities on local causality are restricted to each individual instances (such as photon polarization) but not to the entire population. This is a good illustration that the local causality is likely correct and the quantum mechanics prediction on a unified system is an approximation (not physical).