

Definition of Symmetry of Determinants

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Abstract

Provides a definition of symmetry for determinants, which simplifies the proof of two results for determinants.

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The Symmetry Definition of a Determinant

There are n^2 numbers arranged in a table with n rows
 n columns

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

The a_{ij} is called the element of D , it has two subscripts.

The first subscript i is called the row subscript, and the second subscript j is called the column subscript. The row subscript i and column subscript j indicate that the element is the row i column j element of this D , for example, a_{21} is the second row first column element of D .

Make the n product of the elements located in different rows and different columns in the table D , and label it with a symbol $(-1)^t$, obtain the form as follows

$$(-1)^t a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n} \quad (1)$$

is called the items, which $p_1 p_2 \cdots p_n$ is an arrangement of natural numbers $1, 2, \cdots, n$, $q_1 q_2 \cdots q_n$ is also an arrangement of natural numbers $1, 2, \cdots, n$

$$t = \tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n) \quad \text{where}$$

$\tau(p_1 p_2 \cdots p_n)$ is the inverse order number of the arrangement $p_1 p_2 \cdots p_n$, $\tau(q_1 q_2 \cdots q_n)$ is the inverse order number of the arrangement $q_1 q_2 \cdots q_n$.

Because arrangements $p_1 p_2 \cdots p_n$ have $n!$ items, $q_1 q_2 \cdots q_n$ also have $n!$ items, so items in the form of (1)

have $(n!)^2$ items. Algebraic sum of the $(n!)^2$ items and division by $n!$, i.e

$\frac{1}{n!} \sum (-1)^t a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n} = D$ is called an n

order determinant, denoted as

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$$

$$\frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n}$$

Where, a_{ij} as mentioned earlier, are the (i, j) elements of this determinant D , that is, the elements of the row i and column j of this determinant D .

Assuming $i_1 \cdots i_k \cdots i_l \cdots i_n$ is a fixed arrangement, exchange i_k, i_l obtain a new fixed arrangement $i_1 \cdots i_l \cdots i_k \cdots i_n$, then

$$\begin{aligned} & \sum_{q_1 \cdots q_k \cdots q_l \cdots q_n} (-1)^{\tau(i_1 \cdots i_k \cdots i_l \cdots i_n) + \tau(q_1 q_2 \cdots q_n)} a_{i_1 q_1} \cdots a_{i_k q_k} \cdots a_{i_l q_l} \cdots a_{i_n q_n} \\ &= \sum_{q_1 \cdots q_l \cdots q_k \cdots q_n} (-1)^{\tau(i_1 \cdots i_l \cdots i_k \cdots i_n) + \tau(q_1 q_2 \cdots q_n)} a_{i_1 q_1} \cdots a_{i_l q_l} \cdots a_{i_k q_k} \cdots a_{i_n q_n} \end{aligned}$$

This is because each permutation changes the parity of the arrangement and the multiplication of numbers satisfies the commutative law. Pay attention to that in each of the $n!$ arrangements here, q_k, q_l has been transformed into q_l, q_k , the arrangement $q_1 \cdots q_k \cdots q_l \cdots q_n$ has been transformed into $q_1 \cdots q_l \cdots q_k \cdots q_n$ where the q_k, q_l positions remain unchanged, and their values are swapped and vary with different arrangements! It can be inferred from above that

$$\sum_{q_1 q_2 \cdots q_n} (-1)^{\tau(i_1 \cdots i_k \cdots i_l \cdots i_n) + \tau(q_1 q_2 \cdots q_n)} a_{i_1 q_1} \cdots a_{i_k q_k} \cdots a_{i_l q_l} \cdots a_{i_n q_n}$$

$$= \sum_{q_1 q_2 \cdots q_n} (-1)^{\tau(12 \cdots n) + \tau(q_1 q_2 \cdots q_n)} a_{1 q_1} a_{2 q_2} \cdots a_{n q_n}$$

$$= \sum_{q_1 q_2 \cdots q_n} (-1)^{\tau(q_1 q_2 \cdots q_n)} a_{1 q_1} a_{2 q_2} \cdots a_{n q_n}$$

This is because any n order arrangement can be transformed into $12 \cdots n$ by a finite number of times. Also, because there are $n!$ n -order permutations in total, hence, the new determinant defined in this article is equal to the classical determinant!

This definition can be extended to row subscript as n different natural numbers and column subscript as n different natural numbers. Just arrange these n different natural numbers from small to large, then let them take values $1, 2, \cdots, n$ in sequence!

Example 1 Find the determinant of order n

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

Solution The elements a_{ij} of this determinant can be non-zero only in the case $i = j$, so,

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix} =$$

$$\begin{aligned} & \frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n} \\ &= \frac{1}{n!} \sum_{p_1 p_2 \cdots p_n} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(p_1 p_2 \cdots p_n)} a_{p_1 p_1} a_{p_2 p_2} \cdots a_{p_n p_n} \\ &= \frac{1}{n!} \sum_{p_1 p_2 \cdots p_n} a_{p_1 p_1} a_{p_2 p_2} \cdots a_{p_n p_n} = a_{11} a_{22} \cdots a_{nn} \end{aligned}$$

Example 2 Find the determinant of order n

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Solution When $p_1=1$, q_1 must be 1, otherwise

$a_{p_1 q_1} = 0$, when $p_2=2$, q_2 must be 2, otherwise

$a_{p_2 q_2} = 0$, and so on $p_i = q_i, i=1, 2, \dots, n$. The

following is the same as Example 1.

Example 3 Find the determinant of order n

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix}$$

Solution For this determinant, only the elements on the second diagonal may not be 0, so only it's worth

considering $q_i = n+1-p_i, i=1, 2, \dots, n$. therefore

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} =$$

$$\begin{aligned} & \frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n} \\ &= \frac{1}{n!} \sum_{p_1 p_2 \cdots p_n} (-1)^t a_{p_1 n+1-p_1} a_{p_2 n+1-p_2} \cdots a_{p_n n+1-p_n} \end{aligned}$$

Because each permutation changes the parity of the arrangement and the multiplication of numbers satisfies the commutative law, and any arrangement of order n can be transformed into $12\dots n$ in a finite number of times, therefore

$$t = \tau(12\dots n) + \tau(n(n-1)\cdots 21) = \frac{n(n-1)}{2}$$

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_{1n} a_{2n-1} \cdots a_{n1}$$

Property Determinant is equal to its transpose determinant.

This is a direct result of the symmetry definition of determinants, and also is the best application of this new definition of determinant.

The Expansion Theorem of Determinants by Rows (Columns)

A determinant is equal to the sum of the products of its elements and their corresponding algebraic cofactor in any row (column), that is

$$D = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in} \quad (i=1, 2, \dots, n)$$

$$D = a_{1j}A_{1j} + a_{2j}A_{2j} + \cdots + a_{nj}A_{nj} \quad (j=1, 2, \dots, n)$$

Proof Prove only for rows, because as long as it holds for rows, the above property immediately deduce that it holds for columns as well!

$$\begin{aligned} D &= \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \\ & \frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n} \\ &= \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots 1 \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots 1 \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i 1} \cdots a_{p_n q_n} \\ &+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots 2 \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots 2 \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i 2} \cdots a_{p_n q_n} \\ &+ \cdots \\ &+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots k \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots k \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i k} \cdots a_{p_n q_n} \\ &+ \cdots \\ &+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots n \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots n \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i n} \cdots a_{p_n q_n} \end{aligned}$$

$$\begin{aligned}
& \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots k \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots k \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i k} \cdots a_{p_n q_n} \\
&= \frac{1}{n!} \sum_{\substack{p_1 p_i \cdots p_{i-1} p_{i+1} \cdots p_n \\ k q_1 \cdots q_{i-1} q_{i+1} \cdots q_n}} (-1)^{t_i} a_{p_i k} a_{p_1 q_1} \cdots a_{p_{i-1} q_{i-1}} a_{p_{i+1} q_{i+1}} \cdots a_{p_n q_n} \\
&+ \frac{1}{n!} \sum_{\substack{p_1 p_i p_2 \cdots p_{i-1} p_{i+1} \cdots p_n \\ q_1 k q_2 \cdots q_{i-1} q_{i+1} \cdots q_n}} (-1)^{t_2} a_{p_1 q_1} a_{p_i k} a_{p_2 q_2} \cdots a_{p_{i-1} q_{i-1}} a_{p_{i+1} q_{i+1}} \cdots a_{p_n q_n} \\
&+ \cdots \\
&+ \frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n \\ q_1 q_2 \cdots q_{i-1} k q_{i+1} \cdots q_n}} (-1)^{t_j} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_{i-1} q_{i-1}} a_{p_i k} a_{p_{i+1} q_{i+1}} \\
&\cdots a_{p_n q_n} \\
&+ \cdots \\
&+ \frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_{i-1} p_{i+1} \cdots p_n p_j \\ q_1 q_2 \cdots q_{i-1} q_{i+1} \cdots q_n k}} (-1)^{t_n} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_{i-1} q_{i-1}} a_{p_{i+1} q_{i+1}} \\
&\cdots a_{p_n q_n} a_{p_i k}
\end{aligned}$$
$$\begin{aligned} t_1 &= \tau(p_i p_1 \cdots p_{i-1} p_{i+1} \cdots p_n) \\ &\quad + \tau(k q_1 q_2 \cdots q_{i-1} q_{i+1} \cdots q_n) \\ t_s &= \tau(p_1 \cdots p_i \cdots p_{i-1} p_{i+1} \cdots p_n) \\ &\quad + \tau(q_1 \cdots k \cdots q_{i-1} q_{i+1} \cdots n) \\ s &= 2, \dots, n, i > 1, \end{aligned}$$

Because each permutation changes the parity of the arrangement and the number s of digits from p_1 to p_i is equal to the number of digits from q_1 to k , therefore

$$\begin{aligned}
& \frac{1}{n!} \sum_{\substack{\beta_1 \beta_1 \cdots \beta_{j-1} \beta_{j+1} \cdots \beta_n \\ k q_1 \cdots q_{j-1} q_{j+1} \cdots q_n}} (-1)^{t_j} a_{\beta_1 k} a_{\beta_1 q_1} \cdots a_{\beta_{j-1} q_{j-1}} a_{\beta_{j+1} q_{j+1}} \cdots a_{\beta_n q_n} \\
&= \frac{1}{n!} \sum_{\substack{\beta_1 \beta_1 \beta_2 \cdots \beta_{j-1} \beta_{j+1} \cdots \beta_n \\ q_1 k q_2 \cdots q_{j-1} q_{j+1} \cdots q_n}} (-1)^{t_j} a_{\beta_1 q_1} a_{\beta_1 k} a_{\beta_2 q_2} \cdots a_{\beta_{j-1} q_{j-1}} a_{\beta_{j+1} q_{j+1}} \\
&\cdots a_{\beta_n q_n} \\
&= \cdots = \\
& \frac{1}{n!} \sum_{\substack{\beta_1 \beta_2 \cdots \beta_{j-1} \beta_j \beta_{j+1} \cdots \beta_n \\ q_1 q_2 \cdots q_{j-1} k q_{j+1} \cdots q_n}} (-1)^{t_j} a_{\beta_1 q_1} a_{\beta_2 q_2} \cdots a_{\beta_{j-1} q_{j-1}} a_{\beta_j k} a_{\beta_{j+1} q_{j+1}} \cdots a_{\beta_n q_n} \\
&= \cdots = \\
& \frac{1}{n!} \sum_{\substack{\beta_1 \beta_2 \cdots \beta_{j-1} \beta_j \beta_{j+1} \cdots \beta_n \beta_j \\ q_1 q_2 \cdots q_{j-1} q_{j+1} \cdots q_n k}} (-1)^{t_n} a_{\beta_1 q_1} a_{\beta_2 q_2} \cdots a_{\beta_{j-1} q_{j-1}} a_{\beta_{j+1} q_{j+1}} \cdots a_{\beta_n q_n} \\
&\cdot a_{\beta_j k} \\
&= \frac{1}{n!} a_{\beta_j k} (-1)^{\beta_j + k} \sum_{\substack{\beta_1 \cdots \beta_{j-1} \beta_{j+1} \cdots \beta_n \\ q_1 \cdots q_{j-1} q_{j+1} \cdots q_n}} (-1)^{t_j} a_{\beta_1 q_1} \cdots a_{\beta_{j-1} q_{j-1}} a_{\beta_{j+1} q_{j+1}} \\
&\cdots a_{\beta_n q_n}
\end{aligned}$$

$$\begin{aligned} & \frac{1}{n!} \sum_{\substack{p_1 \cdots p_r \cdots p_n \\ q_1 \cdots k \cdots q_n}} (-1)^{\tau(p_1 \cdots p_r \cdots p_n) + \tau(q_1 \cdots k \cdots q_n)} a_{p_1 q_1} \cdots a_{p_r k} \cdots a_{p_n q_n} \\ &= \frac{n}{n!} a_{p_r k} (-1)^{p_r + k} \sum_{\substack{p_1 \cdots p_{r-1} p_{r+1} \cdots p_n \\ q_1 \cdots q_{r-1} q_{r+1} \cdots q_n}} (-1)^{t_i} a_{p_1 q_1} \cdots a_{p_{r-1} q_{r-1}} a_{p_{r+1} q_{r+1}} \\ & \quad \cdots a_{p_n q_n} \\ &= \frac{1}{(n-1)!} a_{p_r k} (-1)^{p_r + k} (n-1)! M_{p_r k} = a_{p_r k} A_{p_r k} \end{aligned}$$
$$t_i = \tau(p_1 \cdots p_{i-1} p_{i+1} \cdots p_n) + \tau(q_1 \cdots q_{i-1} q_{i+1} \cdots q_n)$$
$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$$

The theorem has been proven!

Reference

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