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Definition of Symmetry of Determinants

Zhou Zhongwang

School of Mathematics and Statistics, Weifang University, Weifang 261061, Shandong, China

***Corresponding author:** Zhou Zhongwang, School of Mathematics and Statistics, Weifang University, Weifang 261061, Shandong, China.

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Abstract

Provides a definition of symmetry for determinants, which simplifies the proof of two results for determinants.

Keywords: Inverse Order Number, Determinant, Transposed Determinant, Cofactor, Algebraic Cofactor.

The Symmetry Definition of a Determinant

There are n^2 numbers arranged in a table with n rows n columns

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

The a_{ij} is called the element of D, it has two subscripts. The first subscript i is called the row subscript, and the second subscript j is called the column subscript. The row subscript i and column subscript j indicate that the element is the row i column j element of this D, for example, a_{21} is the second row first column element of D.

Make the n product of the elements located in different rows and different columns in the table D, and label it with a symbol $(-1)^t$, obtain the form as follows

$$(-1)^t a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n} \qquad (1)$$

is called the items, which $p_1 p_2 \cdots p_n$ is an arrangement of natural numbers $1, 2, \cdots, n$, $q_1 q_2 \cdots q_n$ is also an arrangement of natural numbers $1, 2, \cdots, n$

$$t = \tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)$$

where $\tau(p_1 p_2 \cdots p_n)$ is the inverse order number of the arrangement $p_1 p_2 \cdots p_n$, $\tau(q_1 q_2 \cdots q_n)$ is the inverse order number of the arrangement $q_1 q_2 \cdots q_n$.

Because arrangements $p_1 p_2 \cdots p_n$ have $n!$ items, $q_1 q_2 \cdots q_n$ also have $n!$ items , so items in the form of (1) have $(n!)^2$ items .Algebraic sum of the $(n!)^2$ items and division by $n!$, i.e

$\frac{1}{n!}\sum(-1)^t a_{p_1q_1}a_{p_2q_2}\cdots a_{p_nq_n}=D$ is called an n

order determinant, denoted as

$$D=\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_n \\ q_1q_2\cdots q_n}}(-1)^{\tau(p_1p_2\cdots p_n)+\tau(q_1q_2\cdots q_n)}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_nq_n}$$

Where, a_{ij} as mentioned earlier, are the (i,j) elements of this determinant D, that is, the elements of the row i and column j of this determinant D.

Assuming $i_1\cdots i_k\cdots i_l\cdots i_n$ is a fixed arrangement, exchange i_k, i_l obtain a new fixed arrangement $i_1\cdots i_l\cdots i_k\cdots i_n$, then

$$\sum_{q_1\cdots q_k\cdots q_l\cdots q_n}(-1)^{\tau(i_1\cdots i_k\cdots i_l\cdots i_n)+\tau(q_1q_2\cdots q_n)}a_{i_1q_1}\cdots a_{i_kq_k}\cdots a_{i_lq_l}\cdots a_{i_nq_n}$$

$$=\sum_{q_1\cdots q_l\cdots q_k\cdots q_n}(-1)^{\tau(i_1\cdots i_l\cdots i_k\cdots i_n)+\tau(q_1q_2\cdots q_n)}a_{i_1q_1}\cdots a_{i_lq_l}\cdots a_{i_kq_k}\cdots a_{i_nq_n}$$

This is because each permutation changes the parity of the arrangement and the multiplication of numbers satisfies the commutative law. Pay attention to that in each of the arrangements here, has been transformed into, the arrangement has been transformed into. where the positions remain unchanged, and their values are swapped and vary with different arrangements!. It can be inferred from above that

$$\sum_{q_1q_2\cdots q_n}(-1)^{\tau(i_1\cdots i_k\cdots i_l\cdots i_n)+\tau(q_1q_2\cdots q_n)}a_{i_1q_1}\cdots a_{i_kq_k}\cdots a_{i_lq_l}\cdots a_{i_nq_n}$$

$$=\sum_{q_1q_2\cdots q_n}(-1)^{\tau(12\cdots n)+\tau(q_1q_2\cdots q_n)}a_{1q_1}a_{2q_2}\cdots a_{nq_n}$$

$$=\sum_{q_1q_2\cdots q_n}(-1)^{\tau(q_1q_2\cdots q_n)}a_{1q_1}a_{2q_2}\cdots a_{nq_n}$$

This is because any n order arrangement can be transformed into $12\cdots n$ by a finite number of times. Also, because there are $n!$ $n-order$ permutations in total, hence, the new determinant defined in this article is equal to the classical determinant!

This definition can be extended to row subscript as n different natural numbers and column subscript as n different natural numbers. Just arrange these n different natural numbers from small to large, then let them take values $1,2,\cdots,n$ in sequence!

Example 1 Find the determinant of order n

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}$$

Solution The elements a_{ij} of this determinant can be non-zero only in the case $i=j$, so,

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{vmatrix}=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_n \\ q_1q_2\cdots q_n}}(-1)^{\tau(p_1p_2\cdots p_n)+\tau(q_1q_2\cdots q_n)}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_nq_n}$$

$$=\frac{1}{n!}\sum_{p_1p_2\cdots p_n}(-1)^{\tau(p_1p_2\cdots p_n)+\tau(p_1p_2\cdots p_n)}a_{p_1p_1}a_{p_2p_2}\cdots a_{p_np_n}$$

$$=\frac{1}{n!}\sum_{p_1p_2\cdots p_n}a_{p_1p_1}a_{p_2p_2}\cdots a_{p_np_n}=a_{11}a_{22}\cdots a_{nn}$$

Example 2 Find the determinant of order n

$$\begin{vmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

Solution When $p_1 = 1$, q_1 must be 1, otherwise $a_{p_1 q_1} = 0$, when $p_2 = 2$, q_2 must be 2, otherwise $a_{p_2 q_2} = 0$, and so on $p_i = q_i, i = 1, 2, \cdots, n$,. The following is the same as Example 1.

Example 3 Find the determinant of order n

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2\,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix}$$

Solution For this determinant, only the elements on the second diagonal may not be 0, so only it's worth considering $q_i = n + 1 - p_i, i = 1, 2, \cdots n$. therefore

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2\,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} =$$

$$\frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n}$$

$$= \frac{1}{n!} \sum_{p_1 p_2 \cdots p_n} (-1)^t a_{p_1 n+1-p_1} a_{p_2 n+1-p_2} \cdots a_{p_n n+1-p_n}$$

Because each permutation changes the parity of the arrangement and the multiplication of numbers satisfies the commutative law, and any arrangement of order can be transformed into in a finite number of times, therefore

$$t = \tau(12 \cdots n) + \tau(n(n-1) \cdots 21) = \frac{n(n-1)}{2}$$

$$\begin{vmatrix} 0 & \cdots & 0 & a_{1n} \\ 0 & \cdots & a_{2\,n-1} & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & 0 & 0 \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} a_{1n} a_{2\,n-1} \cdots a_{n1}$$

Property Determinant is equal to its transpose determinant. This is a direct result of the symmetry definition of determinants, and also is the best application of this new definition of determinant.

The Expansion Theorem of Determinants by Rows (Columns)

A determinant is equal to the sum of the products of its elements and their corresponding algebraic cofactor in any row (column), that is

$$D = a_{i1} A_{i1} + a_{i2} A_{i2} + \cdots + a_{in} A_{in} \, (i = 1, 2, \cdots, n)$$

$$D = a_{1j} A_{1j} + a_{2j} A_{2j} + \cdots + a_{nj} A_{nj} \, (j = 1, 2, \cdots, n)$$

Proof Prove only for rows, because as long as it holds for rows, the above property immediately deduce that it holds for columns as well!

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} =$$

$$\frac{1}{n!} \sum_{\substack{p_1 p_2 \cdots p_n \\ q_1 q_2 \cdots q_n}} (-1)^{\tau(p_1 p_2 \cdots p_n) + \tau(q_1 q_2 \cdots q_n)} a_{p_1 q_1} a_{p_2 q_2} \cdots a_{p_n q_n}$$

$$= \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots 1 \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots 1 \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i 1} \cdots a_{p_n q_n}$$

$$+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots 2 \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots 2 \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i 2} \cdots a_{p_n q_n}$$

$$+ \cdots$$

$$+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots k \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots k \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i k} \cdots a_{p_n q_n}$$

$$+ \cdots$$

$$+ \frac{1}{n!} \sum_{\substack{p_1 \cdots p_i \cdots p_n \\ q_1 \cdots n \cdots q_n}} (-1)^{\tau(p_1 \cdots p_i \cdots p_n) + \tau(q_1 \cdots n \cdots q_n)} a_{p_1 q_1} \cdots a_{p_i n} \cdots a_{p_n q_n}$$

Where, the general term,

$$\frac{1}{n!}\sum_{\substack{p_1\cdots p_i\cdots p_n\\ q_1\cdots k\cdots q_n}}(-1)^{\tau(p_1\cdots p_i\cdots p_n)+\tau(q_1\cdots k\cdots q_n)}a_{p_1q_1}\cdots a_{p_ik}\cdots a_{p_nq_n}$$

$$=\frac{1}{n!}\sum_{\substack{p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ kq_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_1}a_{p_ik}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$+\frac{1}{n!}\sum_{\substack{p_1p_ip_2\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1kq_2\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_2}a_{p_1q_1}a_{p_ik}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$+\cdots$$

$$+\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_ip_{i+1}\cdots p_n\\ q_1q_2\cdots q_{i-1}kq_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_ik}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}$$

$$+\cdots$$

$$+\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_np_i\\ q_1q_2\cdots q_{i-1}q_{i+1}\cdots q_nk}}(-1)^{t_n}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}a_{p_ik}$$

where

$$t_1=\tau(p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n)$$
$$+\tau(kq_1q_2\cdots q_{i-1}q_{i+1}\cdots q_n)$$

$$t_s=\tau(p_1\cdots p_i\cdots p_{i-1}p_{i+1}\cdots p_n)$$
$$+\tau(q_1\cdots k\cdots q_{i-1}q_{i+1}\cdots n)$$
$$s=2,\cdots,n,i>1,$$

when $i=1$,the same processing can be applied.

Because each permutation changes the parity of the arrangement and the number s of digits from p_1 to p_i is equal to the number of digits from q_1 to k , therefore

$$(-1)^{t_s}=(-1)^{\tau(p_1\cdots p_i\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots k\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$=(-1)^{\tau(p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(kq_1\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$=(-1)^{p_i-1+k-1}(-1)^{\tau(p_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$=(-1)^{p_i-1+k-1}(-1)^{\tau(p_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots q_{i-1}q_{i+1}\cdots q_n)}$$

$$s=1,2\cdots,n$$

So, the all sub-terms are equal, that is,

$$\frac{1}{n!}\sum_{\substack{p_ip_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ kq_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_1}a_{p_ik}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$=\frac{1}{n!}\sum_{\substack{p_1p_ip_2\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1kq_2\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_2}a_{p_1q_1}a_{p_ik}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}$$

$$=\cdots=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_ip_{i+1}\cdots p_n\\ q_1q_2\cdots q_{i-1}kq_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_ik}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$=\cdots=$$

$$\frac{1}{n!}\sum_{\substack{p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_np_i\\ q_1q_2\cdots q_{i-1}q_{i+1}\cdots q_nk}}(-1)^{t_n}a_{p_1q_1}a_{p_2q_2}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}\cdots a_{p_nq_n}$$

$$\cdot a_{p_ik}$$

$$=\frac{1}{n!}a_{p_ik}(-1)^{p_i+k}\sum_{\substack{p_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots a_{p_nq_n}$$

Hence the general term

$$\frac{1}{n!}\sum_{\substack{p_1\cdots p_i\cdots p_n\\ q_1\cdots k\cdots q_n}}(-1)^{\tau(p_1\cdots p_i\cdots p_n)+\tau(q_1\cdots k\cdots q_n)}a_{p_1q_1}\cdots a_{p_ik}\cdots a_{p_nq_n}$$

$$=\frac{n}{n!}a_{p_ik}(-1)^{p_i+k}\sum_{\substack{p_1\cdots p_{i-1}p_{i+1}\cdots p_n\\ q_1\cdots q_{i-1}q_{i+1}\cdots q_n}}(-1)^{t_i}a_{p_1q_1}\cdots a_{p_{i-1}q_{i-1}}a_{p_{i+1}q_{i+1}}$$

$$\cdots\cdot a_{p_nq_n}$$

$$=\frac{1}{(n-1)!}a_{p_ik}(-1)^{p_i+k}(n-1)!M_{p_ik}=a_{p_ik}A_{p_ik}$$

Where,

$$t_i=\tau(p_1\cdots p_{i-1}p_{i+1}\cdots p_n)+\tau(q_1\cdots q_{i-1}q_{i+1}\cdots q_n)$$

Therefore,

$$D=\begin{vmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\ a_{21}&a_{22}&\cdots&a_{2n}\\ \cdots&\cdots&\cdots&\cdots\\ a_{n1}&a_{n2}&\cdots&a_{nn}\end{vmatrix}=$$

$$D=a_{p_i1}A_{p_i1}+a_{p_i2}A_{p_i2}+\cdots+a_{p_in}A_{p_in}\,(i=1,2,\cdots,n)$$

The theorem has been proven!

Reference

1. Department of Mathematics Tongji University. Linear Algebra [M], Sixth Edition. Beijing: Higher Education Press, 2014.

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