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# Estimates for Modified-Kantorovich Bernstein type Ra-tional Operators based on Weighted Convergence

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***Abstract***

*In this paper, we introduce a new modification of Balázs-Szabados operators. We deduce a recurrence formula and calculate the moments and central moments as main results for giving main theorems for weighted convergence. As we investigate the weighted approximation properties of a new modification of Kantorovich type of Balázs-Szabados operators by using the weighted modulus of continuity and we provide the main convergence result for the weighted estimation of these new operators.*

**Keywords:** $q$ - Balázs-Szabados Operators, Kantorovich Operators, Weighted Modulus of Continuity.

## Introduction

In 1975, Bernstein type rational functions,

$$R_n(f;x)=\frac{1}{(1+a_nx)^n}\sum_{k=0}^{n} f\left(\frac{k}{b_n}\right)\binom{n}{k}(a_nx)^k \quad (n=1,2,\dots)$$

Introduced and investigated by Balázs, see [1]. In this defini-tion, is a real and single-valued function defined on the interval $[0,\infty), a_n$ and are real numbers that have been appropriately cho-sen and are independent of $f$ Seven years later, in 1982, Balázs and Szabados worked together to enhance the estimate in [1] by select-ing appropriate parameters and under some restrictions for $f(x)$ see [2].

Several researchers have recently investigated various q - gen-eralizations of the Ba-lázs-Szabados operators [3-7]. A new Kantorovich-type analogue of the Balázs-Szabados operators is defined by Hamal and Sabancigil in [8] as follows;

$$R^{*}_{n,q}(f,x)=\sum_{k=0}^{n} r_{n,k}(q,x)\int_0^1 f\left(\frac{[k]_q+q^k t}{b_n}\right)d_qt, \qquad (1)$$

where $f:[0,\infty)\to \mathbb{R}$, $q\in(0,1)$, $a_n=[n]_q^{\beta-1}$, $b_n=[n]_q^{\beta}$, $0<\beta\le\frac{2}{3}$,

$n\in \mathbb{N}$, $x\ge 0$,

$$r_{n,k}(q,x)=\frac{1}{(1+a_nx)^n}\begin{bmatrix} n\\ k\end{bmatrix}_q (a_nx)^k\prod_{s=0}^{n-k-1}\left(1+(1-q)[s]_q a_nx\right)$$

When compared to the various analogues presented in (1), these newly developed operators offer certain advantages. The first ad-vantage is that they are positive for all continuous and real value functions without any restriction on The sec-ond advantage is that they may be used to approximate integrable functions as well.

Due to the rise in studies of sequences of linear positive op-erators, with their modifi-cations being most significant in ap-proximation theory, so many researchers have defined and in-vestigated various types of operators. In this paper, we modify Kantorovich-type of the Balázs-Szabados operators by using a parameter greater than 0 replacing within the operators , which maintained the positivity and linearity of the new sequence de-fined in the next section.

Before stating the main result of these operators, we give some notations and con-cepts of calculus.

For each nonnegative integer the analogue of is given by

$$[n]_q=1+q+q^2+\dots+q^{n-2}+q^{n-1}=\begin{cases}\frac{1-q^n}{1-q} & \text{if } 0<q<1\\ n & \text{if } q=1\end{cases},\ [0]_q=0,$$

and factorial coefficient is defined by,

$$[n]_q!=\begin{cases}[n]_q[n-1]_q\dots[1]_q & \text{if } n\in N-\{0\}\\ 1 & \text{if } n=0\end{cases}$$

binomial coefficient is defined as;

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \qquad (k, n \in \mathbb{N}^+ \text{ and } 0 \le k \le n),$$

the analogue of $(x-a)^n$ is defined by the polynomial as following,

$$(x-a)_q^n = \prod_{j=0}^{n-1}(x - q^j a) \quad \text{and} \quad (x-a)_q^0 = 1$$

For the Gauss's binomial formula is given by

$$(x+a)_q^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^{\frac{j(j-1)}{2}} a^j x^{n-j}.$$

The integral (the Jackson integral) of the function is defined by

$$\int_0^b f(t) d_q t = b(1-q) \sum_{j=0}^{\infty} f(bq^j) q^j, \quad 0<q<1, \quad b>0$$

Also, $\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t, \quad 0<a<b.$

More information about q-calculus can be found in [9, 10].

Let $B_2[0,\infty) = \{ f : [0,\infty) \to R : |f(x)| \le M_f (1+x^2) \}$ where $M_f$

where is a constant depend on the function and $f, C_2[0,\infty) = B_2[0,\infty) \cap C[0,\infty)$

$C_2^*[0,\infty) = \left\{ f \in C_2[0,\infty) : \lim_{x\to\infty} \frac{f(x)}{1+x^2} < \infty \right\}$. The norm of any $f \in C_2^*[0,\infty)$ is given by $\|f\|_\rho = \sup_{x \ge 0} \frac{|f(x)|}{\rho(x)}$. The modulus of continuity of on a closed and bounded interval $[0,a],\ a>0$ is defined as follows:

$\omega_b(f,\delta) = \sup_{|t-x|\le\delta} \sup_{x\in[0,b]} |f(t) - f(x)|$. It is obvious that for a function $f \in C_2[0,\infty)$, the modulus of continuity $\omega_b(f,\delta)$ tends to zero $\delta \to 0$.

## Construction of the Operators and the Recurrence Formula

Definition 1. Let $0<q<1$ and $n \in \mathbb{N}$ for $f \in C[0,\infty)$, and for a new modification of Kantorovich -type of Balázs-Szabados operators be introduced as follows:

$$R_{n,q}^{\lambda}(f,x) = \sum_{k=0}^{n} r_{n,k}(q,x) \int_0^1 f\left( \frac{[k]_q + q^k t^{\lambda}}{b_n} \right) d_q t, \qquad \lambda > 0$$

where $b_n = [n]_q^{\beta}, 0<\beta\le\frac{2}{3}, \ n\in\mathbb{N}, x\ge 0, \ b_n = [n]_q^{\beta}, 0<\beta\le\frac{2}{3},$ $n\in\mathbb{N}, x\ge0,$ and

$$r_{n,k}(q,x) = \frac{1}{(1+a_n x)^n} \begin{bmatrix} n \\ k \end{bmatrix}_q (a_n x)^k \prod_{s=0}^{n-k-1} \left(1 + (1-q)[s]_q a_n x\right).$$

For $\lambda = 1$, these polynomials reduce to a new Kantorovich-type analogue of the Balázs-Szabados operators defined by Hamal and Sabancıgil see [8].

In addition, for the results coincide with the results for q-BSK operators defined by Ozkan see [5].

In the next lemma, we give the recurrence formula that is needed for the evaluation of the moments of new modification- Kantorovich type of the B-S operators.

**Lemma 1.** For $n\in\mathbb{N}$, $x\in[0,\infty), m\in\mathbb{N}^+\cup\{0\}$ and $0<q<1$ we have

$$R_{n,q}^{\lambda}(t^m;x) = \sum_{j=0}^{m} \binom{m}{j} \frac{1}{b_n^{m-j}[\lambda(m-j)+1]_q} \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i (q^n-1)^i R_{n,q}(t^{i+j};x) \qquad (3)$$

Proof. By direct calculation, the recurrence formula is obtained as follows:

$$R_{n,q}^{\lambda}(t^m;x) = \sum_{k=0}^{n} r_{n,k}(q;x) \int_0^1 \left( \frac{[k]_q + q^k t^{\lambda}}{b_n} \right)^m d_q t,$$

with the assistance of the binomial formula to $\left([k]_q + q^k t^{\lambda}\right)^m$ and the evaluating of the integral as follows:

$$\left([k]_q + q^k t^{\lambda}\right)^m = \sum_{j=0}^{m} \binom{m}{j} \left([k]_q\right)^j \left(q^k t^{\lambda}\right)^{m-j},$$

$$\int_0^1 t^{\lambda(m-j)} d_q t = (1-q) \sum_{k=0}^{\infty} q^k \cdot q^{\lambda k(m-j)} = \frac{1}{[\lambda(m-j)+1]_q},$$

we get,

$$R_{n,q}^{\lambda}(t^m;x) = \sum_{k=0}^{n} r_{n,k}(q,x) \sum_{j=0}^{m} \binom{m}{j} \frac{q^{(m-j)k}[k]_q^j}{b_n^m [\lambda(m-j)+1]_q}$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[\lambda(m-j)+1]_q} \sum_{k=0}^{n} q^{(m-j)k} \frac{[k]_q^j}{b_n^m} r_{n,k}(q,x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{[\lambda(m-j)+1]_q} \sum_{k=0}^{n} (q^k+1-1)^{m-j} \frac{[k]_q^j}{b_n^m} r_{n,k}(q,x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}[\lambda(m-j)+1]_q} \times \sum_{k=0}^{n} \sum_{i=0}^{m-j} \binom{m-j}{i} (q^k-1)^i \frac{[k]_q^j}{b_n^j} r_{n,k}(q,x)$$

$$= \sum_{j=0}^{m} \binom{m}{j} \frac{1}{(b_n)^{m-j}[\lambda(m-j)+1]_q} \times \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i (q^n-1)^i \sum_{k=0}^{n} \frac{[k]_q^{i+j}}{(b_n)^{i+j}} r_{n,k}(q,x),$$

Since the last summation is $R_{n,q}(t^{i+j};x)$ then we obtain

$$R_{n,q}^{\lambda}(t^m;x) = \sum_{j=0}^{m} \binom{m}{j} \frac{1}{b_n^{m-j}[\lambda(m-j)+1]_q} \sum_{i=0}^{m-j} \binom{m-j}{i} (a_n)^i (q^n-1)^i R_{n,q}(t^{i+j};x).$$

So, $R_{n,q}^{\lambda}(t^m;x)$ is a polynomial of degree less than or equal to $\min(m,n)$ and

$$\frac{1}{b_n^{m-j}[\lambda(m-j)+1]_q} > 0, \ j \in \{0,1,2,...m\}.$$

In the following lemma, we give $R_{n,q}^{\lambda}(f;x)$ for the monomials $f(t) = t^m$, $m = 0,1,2,3.$

**Lemma 2.** For all $n \in \mathbb{N}$, $x \in [0,\infty)$ and $0<q<1$, we have the following ties;

$$R_{n,q}^{\lambda}(1;x)=1,$$

$$R_{n,q}^{\lambda}(t;x)=\frac{2q}{[\lambda+1]_q}\frac{x}{1+a_nx}+\frac{1}{[\lambda+1]_q b_n} \quad (4)$$

$$R_{n,q}^{\lambda}(t^2;x)=\frac{q[n-1]_q}{[n]_q}\frac{4q^3+q^2+q}{[\lambda+1]_q[\lambda+2]_q}\left(\frac{x}{1+a_nx}\right)^2+\frac{4q^3+5q^2+3q}{[\lambda+1]_q[\lambda+2]_q b_n}\left(\frac{x}{1+a_nx}\right)+\frac{1}{[\lambda+2]_q b_n^2} \quad (5)$$

$$R_{n,q}^{\lambda}(t^3;x)=\left(\frac{(q-1)^3}{[\lambda+3]_q}+\frac{3(q-1)^2}{[\lambda+2]_q}+\frac{3(q-1)}{[\lambda+1]_q}+1\right)\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2}\left(\frac{x}{1+a_nx}\right)^3$$
$$+\left\{\frac{(q-1)^2(q^2+q+1)}{[\lambda+3]_q b_n}+\frac{3q(q^2-1)}{[\lambda+2]_q b_n}+\frac{3(q^2+q-1)}{[\lambda+1]_q b_n}+\frac{2+q}{b_n}\right\}\frac{q[n-1]_q}{[n]_q}\left(\frac{x}{1+a_nx}\right)^2+\left\{\frac{q^3-1}{[\lambda+3]_q b_n^2}+\frac{3q^2}{[\lambda+2]_q b_n^2}+\frac{3q}{[\lambda+3]_q b_n^2}+\frac{1}{b_n^2}\right\}\frac{x}{1+a_nx}+\frac{1}{[\lambda+3]_q b_n^3},$$

$$R_{n,q}^{\lambda}(t^4;x)=\frac{q^6[n-1]_q[n-2]_q[n-3]_q}{[n]_q^3}\left\{\frac{(q-1)^4}{[\lambda+4]_q}+\frac{4(q-1)^3}{[\lambda+3]_q}+\frac{6(q-1)^2}{[\lambda+2]_q}+\frac{4(q-1)}{[\lambda+1]_q}+1\right\}\left(\frac{x}{1+a_nx}\right)^4$$
$$+\left\{\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2 b_n}\left(([\lambda+2]_q+[\lambda+1]_q+1)+\frac{4([4]_q-3)}{[\lambda+1]_q}+\frac{6(q-1)([\lambda+3]_q-2)}{[\lambda+2]_q}\right)+\frac{q^3[n-1]_q[n-2]_q}{[n]_q^2 b_n}\left(\frac{4[\lambda+2]_q q(q-1)^2}{[\lambda+3]_q}+\frac{[\lambda+3]_q(q-1)^3}{[\lambda+4]_q}\right)\right\}\left(\frac{x}{1+a_nx}\right)^3$$
$$+\left\{\frac{q[n-1]_q(3+3q+q^2)}{[n]_q b_n^2}+\frac{4q[n-1]_q(q^3+2q^2+q+1)}{[\lambda+1]_q[n]_q b_n^2}+\frac{6q^2[n-1]_q((q-1)^3+2(q-1)+q^3)}{[\lambda+2]_q[n]_q b_n^2}+\frac{4q^3[n-1]_q(q^3-1)}{[\lambda+3]_q[n]_q b_n^2}\right.$$

## Main Results

By using the linearity property of the operators and Lemma 2, we obtain the central moments of these new operators and at the same time, we give their esti-mations in the following Lemma.

Lemma 3. For all $n \in \mathbb{N}$ and $0<q<1$, we have the following estimations;

$$\left(R_{n,q}^{\lambda}((t-x);x)\right)^2\le\frac{2}{\left(b_n[\lambda+1]_q\right)^2}\left\{\left([n]_q\left([\lambda+1]_q(1+a_nx)-2q\right)\right)^2+1\right\},\ x\in[0,\infty)$$

$$R_{n,q}^{\lambda}\left((t-x)^2;x\right)\le\frac{2}{b_n}\left\{\frac{1}{[\lambda+2]_q b_n}+\frac{\left(x+a_n^2b_nx^4\right)}{(1+a_nx)^2}\right\},\ x\in[0,\infty)$$

**Proof**

By using previous lemma and linearity of $R_{n,q}^{\lambda}$, we have

$$\left(R_{n,q}^{\lambda}((t-x);x)\right)^2=\left(\frac{2q}{[\lambda+1]_q}\frac{x}{1+a_nx}-x+\frac{1}{[\lambda+1]_q b_n}\right)^2$$

$$\le 2\left(\frac{2q}{[\lambda+1]_q}\frac{x}{1+a_nx}-x\right)^2+2\left(\frac{1}{[\lambda+1]_q b_n}\right)^2$$

$$\le 2\left(\frac{\left(2q-[\lambda+1]_q\right)x}{[\lambda+1]_q(1+a_nx)}\frac{x}{1+a_nx}-x\right)^2+2\left(\frac{1}{[\lambda+1]_q b_n}\right)^2$$

$$\le 2\left(\frac{b_n}{b_n}\frac{\left([\lambda]_q+q^{\lambda}-2q\right)x}{[\lambda+1]_q(1+a_nx)}+\frac{a_nx^2}{1+a_nx}\right)^2+2\left(\frac{1}{[\lambda+1]_q b_n}\right)^2$$

$$\le 2\left(\frac{[n]_q}{b_n}\frac{\left([\lambda]_q+q^{\lambda}-2q\right)}{[\lambda+1]_q}\frac{a_nx}{(1+a_nx)}+\frac{b_n}{b_n}\frac{a_nx^2}{1+a_nx}\right)^2+2\left(\frac{1}{[\lambda+1]_q b_n}\right)^2$$

$$\le 2\left(\frac{[n]_q}{b_n}\frac{\left([\lambda]_q+q^{\lambda}-2q\right)}{[\lambda+1]_q}\frac{a_nx}{(1+a_nx)}+\frac{b_n}{b_n}\frac{a_nx^2}{1+a_nx}\right)^2+2\left(\frac{1}{[\lambda+1]_q b_n}\right)^2$$

$$\le\frac{2}{\left(b_n[\lambda+1]_q\right)^2}\left\{\left(\frac{1-q^n}{1-q}\left([\lambda]_q+q^{\lambda}-2q\right)+\frac{[\lambda+1]_q(1-q^n)}{1-q}a_nx\right)^2+1\right\}$$

$$\le\frac{2}{\left(b_n[\lambda+1]_q\right)^2}\left\{\left([n]_q\left([\lambda]_q+q^{\lambda}-2q\right)+[\lambda+1]_q a_nx\right)^2+1\right\}.$$

For the estimation of, $R_{n,q}^{\lambda}\left((t-x)^2;x\right)$,

$$R_{n,q}^{\lambda}\left((t-x)^2;x\right)=\sum_{k=0}^{n}r_{n,k}(q,x)\int_0^1\left(\frac{[k]_q+q^kt^{\lambda}}{b_n}-x\right)^2d_qt$$

$$=\sum_{k=0}^{n}r_{n,k}(q,x)\int_0^1\left(\frac{q^kt^{\lambda}}{b_n}+\frac{[k]_q}{b_n}-x\right)^2d_qt$$

$$\le 2\sum_{k=0}^{n}r_{n,k}(q,x)\int_0^1\left(\frac{q^kt^{\lambda}}{b_n}\right)^2d_qt+2\sum_{k=0}^{n}r_{n,k}(q,x)\int_0^1\left(\frac{[k]_q}{b_n}-x\right)^2d_qt$$

$$\le 2\sum_{k=0}^{n}r_{n,k}(q,x)\frac{q^{2k}}{b_n^2}\int_0^1t^{2\lambda}d_qt+2\sum_{k=0}^{n}r_{n,k}(q,x)\int_0^1\left(\frac{[k]_q}{b_n}-x\right)^2d_qt$$

$$\le 2\sum_{k=0}^{n}r_{n,k}(q,x)\frac{q^{2k}}{[\lambda+2]_q b_n^2}++2\sum_{k=0}^{n}r_{n,k}(q,x)\left(\frac{[k]_q}{b_n}-x\right)^2$$

Now, by using formula of $R_{n,q}\left((t-x)^2;x\right)$, so get

$$R_{n,q}^{\lambda}\left((t-x)^2;x\right)\le\frac{2}{[\lambda+2]_q b_n^2}+2\left(\frac{x+a_n^2b_nx^4}{b_n(1+a_nx)^2}\right).$$

Theorem 1. Assume that $q=q_n$ satisfies $0<q_n<1$. For each converges to uniformly on if and only if

$R_{n,q}^{\lambda}(f,x)$ converges to $f$ uniformly on $[0,a]$, $a>0$ if and only if $q_n\to1$ as $n\to\infty$.

**Proof.** Assume that $q_n\to1$ as $n\to\infty$ and $a$ be a fixed greater than 1.

We consider the lattice homeomorphism $T_a:C[0,\infty)\to C[0,a]$ defined by

$T_a(f) = f_{[0,a]}$, it is obvious that

$$T_a\left(R_{n,q}^{\lambda}(1,x)\right) \to T_a(1),\quad T_a\left(R_{n,q}^{\lambda}(t,x)\right) \to T_a(t),\quad T_a\left(R_{n,q}^{\lambda}(t^2,x)\right) \to T_a(t^2)$$

uniformly continuous on $[0,a]$, so we can see that $C_2^*[0,\infty)$ is isomorphic to $C[0,1]$ and the set $\{1,t,t^2\}$ is a Korovkin set in $C_2^*[0,\infty)$. Therefore, according to the known-Korovkin property (Them 4.1.4 in [1], we have $\lim\limits_{n\to\infty} R_{n,q_n}^{\lambda}(f,x) = f(x)$ uniformly on $[0,a]$. We assume that by contradiction $q_n$ doesn't converge to 1 as $n\to\infty$. Then there exist subsequence $q_{n_l} \in (0,1)$ such that $\lim\limits_{n\to\infty} q_{n_l} = c \in [0,1)$, and is it clear that $q_{n_l}^{n_l} \to 1$ as $n\to\infty$. Then from equation (4) in Lemma 2, we have

$$R_{n,q}^{\lambda}(t;x) - x = \frac{2q_{n_l}}{[\lambda+1]_{q_{n_l}}}\frac{x}{1+a_{q_{n_l}}x} + \frac{1}{[\lambda+1]_{q_{n_l}} b_{q_{n_l}}} - x, \text{ for } x\in[0,\infty)$$

We may see,

$$R_{n,q}^{\lambda}(t;x) - x \to \frac{2c}{[\lambda+1]_{q_{n_l}}} \not\to 0 \quad \text{as} \quad n\to\infty.$$

This implies to contradiction to $\lim\limits_{n\to\infty} R_{n,q_n}^{\lambda}(f,x) = f(x)$ uniformly continuous on compact set on $[0,\infty)$, that is the result desired.

**Theorem 2.** Let $0<q_n<1$, $f\in C_2[0,\infty)$, $a>0$ and

$$\omega_{a+1}(f,\delta) = \left\{ \sup_{x,t\in[0,a]} |f(t)-f(x)| : |t-x|\le\delta \right\}$$ be the modulus of continuity on $[0,a+1]\subset[0,\infty)$, then we have

$$\left\|R_{n,q_n}^{\lambda}(f,x) - f(x)\right\|_{[0,a]} \le \left|R_{n,q_n}^{\lambda}(f,x) - f(x)\right| \le L\eta_n(a) + \omega_{a+1}(f,\delta),$$

where $\eta_n(a) = \dfrac{2}{b_{n,q_n}}\phi(q_n,a)$, $\phi(q_n,a) = \left\{ \dfrac{1}{[\lambda+2]_q b_n} + \dfrac{(x+a_n^2 b_n x^4)}{(1+a_n x)^2} \right\}$

and $L = 4M_f(1+a^2)$.,

**Proof.** For $x\in[0,a]$ and $t>b+1$, since $t-x>1$, we have

$$|f(t)-f(x)| \le M_f(2+x^2+t^2)$$

$$\le M_f\left(2(t-x)^2 + 3x^2(t-x)^2 + 2(t-x)^2\right)$$

$$\le M_f(4+3x^2)(t-x)^2$$

$$\le 4M_f(1+a^2)(t-x)^2 \qquad (6)$$

for $x\in[0,a]$, $t<a+1$, we have

$$|f(t)-f(x)| \le \omega_{a+1}(f,|t-x|)$$

$$\le \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta) \qquad (7)$$

So, with $\delta>0$, $x\in[0,b]$, $t\ge 0$ and by the inequalities (6) and (7) we may write

$$|f(t)-f(x)| \le 4M_f(1+a^2)(t-x)^2 + \left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),$$

by applying $R_{n,q_n}^{\lambda}$ to the above inequality and by the well-known Cauchy-Schwarz Inequality, we obtain

$$\left|R_{n,q_n}^{\lambda}(f,x) - f(x)\right| \le 4M_f(1+b^2)R_{n,q_n}^{\lambda}\left((t-x)^2,x\right) + \left(1+R_{n,q_n}^{\lambda}\left(\frac{|t-x|}{\delta},x\right)\right)\omega_{a+1}(f,\delta),$$

$$\left|R_{n,q_n}^{\lambda}(f,x) - f(x)\right| \le 4M_f(1+a^2)R_{n,q_n}^{\lambda}\left((t-x)^2,x\right) + \left(1+\frac{1}{\delta}\left(R_{n,q_n}^{\lambda}\left((t-x)^2,x\right)\right)^{\frac{1}{2}}\right)\omega_{a+1}(f,\delta),$$

by using first result of estimation in Lemma 3,

$$R_{n,q_n}^{\lambda}\left((t-x)^2,x\right) \le \frac{2}{b_{n,q_n}}\left\{ \frac{1}{[\lambda+2]_{q_n} b_{n,q_n}} + \frac{(x+a_{n,q_n}^2 b_{n,q_n} x^4)}{(1+a_{n,q_n}x)^2} \right\}, \text{ for } x\in[0,a)$$

as a result,

$$\left|R_{n,q_n}^{\lambda}(f,x) - f(x)\right| \le 4M_f(1+a^2)\eta_n(a) + \left(1+\frac{1}{\delta}(\eta_n(a))^{\frac{1}{2}}\right)\omega_{a+1}(f,\delta),$$

now, by taking $\delta = \sqrt{\eta_n(a)}$ we get the desired result.

## Weighted Approximation

Let $B_\sigma(\mathbb{R}^+)$ be a weighted space of functions $f(x)$ defined on $\mathbb{R}^+ = [0,\infty)$ and satisfy the inequality $|f(x)| \le L_f\sigma(x)$, where $\sigma(x)$ represents a weighted function that is continuously increasing on $\mathbb{R}^+ = [0,\infty)$, $\sigma(x)\ge 1$ and $L_f$ represents a positive constant depending on $f$. The norm of each function $f\in B_\sigma(\mathbb{R}^+)$ is given by $\|f\|_\sigma = \sup\limits_{x\in\mathbb{R}^+}\dfrac{|f(x)|}{\sigma(x)}$. We consider the following spaces:

$$C_\sigma[0,\infty) = \{f : f\in B_\sigma[0,\infty) \text{ and } f \text{ is continuous}\},$$

$$C_\sigma^*[0,\infty) = \left\{ f : f\in C_\sigma[0,\infty) \text{ and } \lim_{x\to\infty}\frac{f(x)}{\sigma(x)} < \infty \right\}.$$

**Remark 1.** Let $\sigma(x)$ be a weighted function such that $\sigma(x)\ge 1$, and the below inequality $\left|R_{n,q}^{\lambda}(\sigma,x)\right| \le L\sigma(x)$, $L>0$, is satisfied. Then we can say that the sequence of positive linear operators $\left(R_{n,q}^{\lambda}\right)_{n\ge 1}$ acts from $C_\sigma[0,\infty)$ to $B_\sigma[0,\infty)$, for more information see [14].

In the next theorem we investigate weighted approximation theorem for $\left\{R_{n,q}^{\lambda}\right\}_{n\in\mathbb{N}}$.

**Theorem 3.** Assume that $q = q_n$, satisfies $0<q_n<1$ and $q_n\to 1$ as $n\to\infty$. Then for each function $f\in C_2^*[0,\infty)$, we have

$$\lim_{n\to\infty}\left\|R_{n,q_n}^{\lambda}(f;x) - f(x)\right\|_{\rho(x)} = 0$$

**Proof.** By using the Korovkin type on weighted approximation in [12], it is sufficient to satisfy

$$\lim_{n\to\infty}\left\|R_{n,q_n}^{\lambda}(t^m;x) - x^m\right\|_{\rho(x)} = 0, \text{ for } m=0,\ 1,\ 2. \qquad (8)$$

Since $R_{n,q_n}^{\lambda}(1;x) = 1$, holds for $m=0$. By helping of Lemma 2, we have

$$R_{n,q_n}^{\lambda}(t;x) - x = \frac{1}{[\lambda+1]_{q_n} b_{n,q_n}} + \frac{2q_n}{[\lambda+1]_{q_n}}\frac{x}{1+a_{n,q_n}x} - x$$

$$= \frac{1}{[\lambda+1]_{q_n} b_{n,q_n}} + \frac{\left(2q_n - [\lambda+1]_{q_n}\right)x}{1+a_{n,q_n}x} - \left(\frac{a_{n,q_n}x}{1+a_{n,q_n}x}\right)x$$

By applying triangle inequality, we have

$$\left|R_{n,q_n}^{\lambda}(t;x) - x\right| \le \frac{1}{[\lambda+1]_{q_n} b_{n,q_n}} + \frac{\left([\lambda+1]_{q_n} - 2q_n\right)x}{[\lambda+1]_{q_n}(1+a_{n,q_n}x)} + \frac{a_{n,q_n}x^2}{(1+a_{n,q_n}x)}$$

For $\rho(x) = 1+x^2$, we obtain

$$\left\|R_{n,q_n}^{\lambda}(t;x) - x\right\|_{\rho(x)} \le \sup_{0\le x<\infty}\frac{1}{\rho(x)}\left\{ \frac{1}{[\lambda+1]_{q_n} b_{n,q_n}} + \frac{\left([\lambda+1]_{q_n} - 2q_n\right)x}{[\lambda+1]_{q_n}(1+a_{n,q_n}x)} + \frac{a_{n,q_n}x^2}{(1+a_{n,q_n}x)} \right\}$$

$$\leq\left\{\frac{1}{[\lambda+1]_{q_n} b_{n,q_n}}\sup_{0\leq x<\infty}\frac{1}{\rho(x)}+\frac{\left([\lambda+1]_{q_n}-2q_n\right)}{[\lambda+1]_{q_n}}\sup_{0\leq x<\infty}\frac{x}{\rho(x)\left(1+a_{n,q_n}x\right)}\right.$$

$$\left.+a_{n,q_n}\sup_{0\leq x<\infty}\frac{x^2}{\rho(x)\left(1+a_{n,q_n}x\right)}\right\},$$

now by taking the limit overall last inequality, we have

$$\lim_{n\to\infty}\left\|R^*_{n,q_n}(t;x)-x\right\|_2\leq\lim_{n\to\infty}\frac{1}{[\lambda+1]_{q_n} b_{n,q_n}}+\lim_{n\to\infty}\frac{(1-q_n)}{[\lambda+1]_{q_n}}+\lim_{n\to\infty}a_{n,q_n}=0$$

As a result, we get

$$\lim_{n\to\infty}\left\|R^\lambda_{n,q_n}(t;x)-x\right\|_{\rho(x)}=0.$$

Again, by using lemma 2, we have

$$R^*_{n,q_n,\lambda}\left(t^2;x\right)-x^2=\frac{1}{[\lambda+2]_{q_n} b^2_{n,q_n}}+\frac{4q_n^3+5q_n^2+3q_n}{b_n[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\frac{x}{1+a_{n,q_n}x}$$

$$+\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}\frac{4q_n^3+q_n^2+q_n}{[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\frac{x^2}{\left(1+a_{n,q_n}x\right)^2}-x^2$$

$$=\frac{1}{[\lambda+2]_{q_n} b^2_{n,q_n}}+\frac{4q_n^3+5q_n^2+3q_n}{b_n[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\frac{x}{1+a_{n,q_n}x}$$

$$+\left\{\frac{q_n[n-1]_{q_n}}{[n]_{q_n}}\frac{4q_n^3+q_n^2+q_n}{[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\frac{1}{\left(1+a_{n,q_n}x\right)^2}-1\right\}x^2$$

Therefore,

$$\left|R^*_{n,q_n,\lambda}\left(t^2;x\right)-x^2\right|\leq\left\{\frac{1}{[\lambda+2]_{q_n} b^2_{n,q_n}}+\frac{4q_n^3+5q_n^2+3q_n}{b_n[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\frac{x}{1+a_{n,q_n}x}+\right.$$

$$\left.+\frac{4q_n^3+q_n^2+q_n}{[n]_{q_n}[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\frac{x^2}{\left(1+a_{n,q_n}x\right)^2}\right\}$$

$$+\left\{1-\frac{4q_n^3+q_n^2+q_n}{[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\frac{1}{\left(1+a_{n,q_n}x\right)^2}\right\}x^2$$

Then, we have

$$\left\|R^*_{n,q_n,\lambda}\left(t^2;x\right)-x^2\right\|_2\leq\frac{1}{[\lambda+2]_{q_n} b^2_{n,q_n}}\sup_{0\leq x<\infty}\frac{1}{1+x^2}$$

$$+\frac{4q_n^3+5q_n^2+3q_n}{b_n[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\sup_{0\leq x<\infty}\frac{x}{\left(1+a_{n,q_n}x\right)\left(1+x^2\right)}+\frac{4q_n^3+q_n^2+q_n}{[n]_{q_n}[\lambda+1]_{q_n}[\lambda+2]_{q_n}}$$

$$\times\sup_{0\leq x<\infty}\frac{x^2}{\left(1+a_{n,q_n}x\right)^2\left(1+x^2\right)}+\sup_{0\leq x<\infty}\frac{x^2}{\left(1+a_{n,q_n}x\right)^2\left(1+x^2\right)}$$

$$-\frac{4q_n^3+q_n^2+q_n}{[\lambda+1]_{q_n}[\lambda+2]_{q_n}}\sup_{0\leq x<\infty}\frac{x^2}{\left(1+a_{n,q_n}x\right)^2\left(1+x^2\right)}.$$

Now by taking the limit overall last inequality, we have

$$\lim_{n\to\infty}\left\|R^\lambda_{n,q_n}\left(t^2;x\right)-x^2\right\|_2\leq\lim_{n\to\infty}\frac{1}{[\lambda+2]_{q_n} b^2_{n,q_n}}+\lim_{n\to\infty}\frac{4q_n^3+5q_n^2+3q_n}{b_n[1+\lambda]_{q_n}[\lambda+2]_{q_n}}$$

$$+\lim_{n\to\infty}\frac{4q_n^3+q_n^2+q_n}{[n]_{q_n}[1+\lambda]_{q_n}[\lambda+2]_{q_n}}+1-\lim_{n\to\infty}\frac{4q_n^3+q_n^2+q_n}{[1+\lambda]_{q_n}[\lambda+2]_{q_n}}.$$

Hence,

$$\lim_{n\to\infty}\left\|R^\lambda_{n,q_n}\left(t^2;x\right)-x^2\right\|_{\rho(x)=1+x^2}=0.$$

Now, we present the next theorem to approximate all functions in $C_2^*[0,\infty)$. This type of result is given for locally integrable functions. These types of results are given in [15].

**Theorem 4.** Let $0<q_n<1$, $q_n\to 1$ as $n\to\infty$. Then for each $C_2^*[0,\infty)$ and all $\upsilon>0$, we can obtain

$$\lim_{x\to\infty}\sup_{x\in[0,\infty)}\frac{\left|R^\lambda_{n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}=0$$

**Proof.** Let $x_0\in(0,\infty)$ be arbitrary as a fixed. Then

$$\sup_{x\in[0,\infty)}\frac{\left|R^\lambda_{n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}=\sup_{x\leq x_0}\frac{\left|R^\lambda_{n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}+\sup_{x>x_0}\frac{\left|R^\lambda_{n,q_n}(f;x)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}$$

$$\leq\left\|R^\lambda_{n,q_n}(f;x)-f(x)\right\|_{C[0,x_0]}+\sup_{x\in[0,\infty)}\frac{\left|R^\lambda_{n,q_n}\left(\left(1+t^2\right)f;x\right)-f(x)\right|}{\left(1+x^2\right)^{1+\upsilon}}\quad(9)$$

Now, by definition of the norm of each function in $C_2^*[0,\infty)$, we have

$|f(x)|\leq\|f\|_2\left(1+x^2\right)$, also, we have $\sup_{x>x_0}\frac{|f(x)|}{\left(1+x^2\right)^{1+\upsilon}}\leq\frac{\|f\|_2}{\left(1+x^2\right)^{\upsilon}}\leq\frac{\|f\|_2}{\left(1+x_0^2\right)^{\upsilon}}$

Then, let $\varepsilon>0$ be an arbitrary. We can choose $x_0$ to be large that

$$\frac{\|f\|_2}{\left(1+x_0^2\right)^{\upsilon}}<\frac{\varepsilon}{3}.\quad(10)$$

By theorem 1, we get

$$\|f\|_2\lim_{n\to\infty}\frac{\left|R^\lambda_{n,q_n}\left(\left(1+t^2\right);x\right)\right|}{\left(1+x^2\right)^{1+\upsilon}}=\frac{1+x^2}{\left(1+x^2\right)^{1+\upsilon}}\|f\|_2\leq\frac{\|f\|_2}{\left(1+x^2\right)^{\upsilon}}\leq\frac{\|f\|_2}{\left(1+x_0^2\right)^{\upsilon}}<\frac{\varepsilon}{3}.$$

Using of theorem 5.7 page 168-169 in [15], we can see that the first term of the inequality (9)
implies that

$$\left\|R^\lambda_{n,q_n}(f;x)-f(x)\right\|_{C[0,x_0]}<\frac{\varepsilon}{3},\quad\text{as } n\to\infty\quad(11)$$

By taking limit over inequality (9) and combining (10) and (11), we get the result.

## Conclusions

In this paper, by using the notions of calculus and weighted modulus of continui-ty we study weighted approximation properties of new modification of the Ba-lázs-Szabados operators. We provided a recurrence relation for these new operators, and then used this recurrence relation to establish the moments up to the fourth order, as well as estimate the central moments. We discuss the rate of convergence for these operators and we give a Korovkin type theorem for weighted approximation.

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