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Energy Estimation of the Moving Particles in 2-Body and 3-Body Problems of Classical Electrodynamics

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Abstract

In previous papers we have derived equations of motion with radiation terms and proved the existence-uniqueness of periodic solution of 2-body and 3-body problems of classical electrodynamics. The number of equations is more than the number of unknown functions – 8 in number for 2-body and 12 in number for 3-body problems. We have proved that two equations in the first case and three equations in the second one are consequences of the rest ones. These equations are used to determine the energy of moving particles (electrons). The main goal of the present paper is to estimate the energy of the moving electrons for Hydrogen-atom, Hydrogen-like atoms and Helium-atom.

Keywords: Classical Electrodynamics, 2- and 3-Body Problems, Energy Estimation of the Moving Particles

Introduction

The main purpose of the present paper is to continue the investigation of 2-body and 3-body problems of classical electrodynamics [1]-[6]. Since the equations of motion are introduced in the Minkowski space they are an overdetermined system. The number of equations of motion for 2 bodies are 8 in number, while for 3 bodies – 12 in number. The unknown functions in the first case are 6, while in the second one – 9. We have proved that the 4-th and 8-th equations for 2-body problem are consequence of the rest ones. For 3 bodies the 4-th, 8-th and 12-th equations are consequence of the rest ones and in this way, we obtain systems with number of equations equal to the number of the unknown functions. From these equations we get expressions for energy because on the right-hand sides of these equations there are only known functions. Using these equations, we estimate the energy of moving particles circling the nuclei for 2- and 3-body atoms.

The paper consists of seven sections and conclusion. Section 1 is an introduction. In Section 2, a derivation of the explicit form of the energy equation is presented, passing to the Euclidean coordinates and introducing delays depending on the unknown trajectories. In Section 3, the Kepler formulation of the energy terms for 2- and 3-body problems are given. In the case of two bodies the equation is only one, while for three bodies – two. In Section 4 we obtain the energy equation in spherical coordinates and give some estimations. In Section 5 we get inequalities which allow us to obtain intervals for the values of energy of the moving particles. Section 6 contains numerical results for Hydrogen, Hydrogen-like and Helium-atoms. Section 7 is a Conclusion.

Recall that the equations of motion for 3-body problem introduced in [3] are:

$$m_1 \frac{d\lambda_r^{(1)}}{ds_1} = \frac{e_1}{c^2}\left(F_{rl}^{(12)}\lambda_l^{(1)} + F_{rl}^{(13)}\lambda_l^{(1)} + \frac{1}{2}\left[\frac{\partial A_l^{(1)ret}}{\partial x_r^{(1)ret}} - \frac{\partial A_r^{(1)ret}}{\partial x_l^{(1)ret}} - \left(\frac{\partial A_l^{(1)adv}}{\partial x_r^{(1)adv}} - \frac{\partial A_r^{(1)adv}}{\partial x_l^{(1)adv}} \right) \right] \lambda_l^{(1)} \right),$$

$$m_2 \frac{d\lambda_r^{(2)}}{ds_2} = \frac{e_2}{c^2}\left(F_{rl}^{(21)}\lambda_l^{(2)} + F_{rl}^{(23)}\lambda_l^{(2)} + \frac{1}{2}\left[\frac{\partial A_l^{(2)ret}}{\partial x_r^{(2)ret}} - \frac{\partial A_r^{(2)ret}}{\partial x_l^{(2)ret}} - \left(\frac{\partial A_l^{(2)adv}}{\partial x_r^{(2)adv}} - \frac{\partial A_r^{(2)adv}}{\partial x_l^{(2)adv}} \right) \right] \lambda_l^{(2)} \right),$$

$$m_3 \frac{d\lambda_r^{(3)}}{ds_3} = \frac{e_3}{c^2}\left(F_{rl}^{(31)}\lambda_l^{(3)} + F_{rl}^{(32)}\lambda_l^{(3)} + \frac{1}{2}\left[\frac{\partial A_l^{(3)ret}}{\partial x_r^{(3)ret}} - \frac{\partial A_r^{(3)ret}}{\partial x_l^{(3)ret}} - \left(\frac{\partial A_l^{(3)adv}}{\partial x_r^{(3)adv}} - \frac{\partial A_r^{(3)adv}}{\partial x_l^{(3)adv}} \right) \right] \lambda_l^{(3)} \right).$$

The equations of motion for 2-body problem are a particular case of the above ones.

As usually, the Latin indices run from 1 to 4, while the Greek – from 1 to 3 (cf. [7]). We establish that the intervals for natural parameters of the electron motion contain the values obtained from quantum mechanics [8]- [10].

Derivation the Explicit Form of the Energy Equation

Recall that we denote by $\left(x_1^{(k)}(t), x_2^{(k)}(t), x_3^{(k)}(t), x_4^{(k)} = ict\right), (k=1,2,3)$ the space-time coordinates of the charged particles. The dot product in the Minkowski space is $\langle a,b\rangle_4 = a_r b_r = \sum_{r=1}^{4} a_r b_r$, while in the 3-dimensional Eucleadian subspace — $\langle a,b\rangle = a_\alpha b_\alpha = \sum_{\alpha=1}^{3} a_\alpha b_\alpha$; c is the vacuum speed of the light; $m_k (k=1,2,3)$ are the proper masses of the particles; $e_k (k=1,2,3)$ — their charges. The elements of proper times are ds_k (k-1,2,3) and the unit tangent vectors to the world lines are $\lambda^{(k)} = \left(\lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_3^{(k)}, \lambda_4^{(k)}\right)$, $\vec{u}^{(k)} = \left(u_1^{(k)}(t), u_2^{(k)}(t), u_3^{(k)}(t)\right) = \left(\dot{x}_1^{(k)}(t), \dot{x}_2^{(k)}(t), \dot{x}_3^{(k)}(t)\right)$,

$$\Delta_k = \sqrt{c^2 - \left\langle \vec{u}^{(k)}(t), \vec{u}^{(k)}(t)\right\rangle}\,;\; \frac{d}{ds_k} = \frac{1}{\Delta_k}\frac{d}{dt}\,;\; \lambda_\alpha^{(k)} = \frac{u_\alpha^{(k)}(t)}{\Delta_k} (\alpha = 1,2,3)\,;\; \lambda_4^{(k)} = \frac{ic}{\Delta_k}\,.$$

The accelerations are $\frac{d\lambda^{(k)}}{ds_k} = \left(\frac{1}{\Delta_k}\frac{d}{dt}\frac{u_\alpha^{(k)}(t)}{\Delta_k}, \frac{ic}{\Delta_k}\frac{d}{dt}\frac{1}{\Delta_k}\right)$. The isotropic 4-vectors

$$\xi^{(kn)} = \left(\xi_1^{(kn)}, \xi_2^{(kn)}, \xi_3^{(kn)}, \xi_4^{(kn)}\right) = \left(x_1^{(k)}(t) - x_1^{(n)}(t-\tau_{kn}), x_2^{(k)}(t) - x_2^{(n)}(t-\tau_{kn}), x_3^{(k)}(t) - x_{13}^{(n)}(t-\tau_{kn}), ic\tau_{kn}\right)$$

$(k=1,2,3; n\neq k)$ satisfy the equalities $\left\langle \xi^{(kn)}, \xi^{(kn)}\right\rangle_4 = 0$ which imply the functional equations for defining $\tau_{kn}(t)$ (6 in number)

$$\tau_{kn}(t) = \frac{1}{c}\sqrt{\left\langle \xi^{(kn)}, \xi^{(kn)}\right\rangle_4} \equiv \frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(k)}(t) - x_\alpha^{(n)}(t-\tau_{kn}(t)\right]^2}\,.$$

For the velocity vectors we get

$$\lambda^{(n)} = \left(\frac{u_1^{(n)}(t-\tau_{kn})}{c}, \frac{u_2^{(n)}(t-\tau_{kn})}{c}, \frac{u_3^{(n)}(t-\tau_{kn})}{c}, \frac{ic}{\Delta_{kn}}\right) = \left(\frac{\vec{u}^{(n)}(t-\tau_{kn})}{c}, \frac{ic}{\Delta_{kn}}\right),$$

where $\Delta_{kn} = \sqrt{c^2 - \left\langle \vec{u}^{(n)}(t-\tau_{kn}), \vec{u}^{(n)}(t-\tau_{kn})\right\rangle}$; $\frac{d}{ds_n} = \frac{1}{\Delta_{kn}}\frac{d}{dt_{kn}} = \frac{1}{\Delta_{kn}}\frac{dt}{dt_{kn}}\frac{d}{dt}$, $\frac{dt}{dt_{kn}} = \frac{c^2\tau_{kn} - \left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)}\right\rangle}{c^2\tau_{kn} - \left\langle \vec{\xi}^{(kn)}, \vec{u}^{(k)}\right\rangle} \equiv D_{kn}$.

In the present paper we investigate only the 4-th and 8-th equations for the 2-body problem

$$\frac{d\lambda_4^{(k)}}{ds_k} = \frac{e_k e_n}{m_k c^2}\left\{\frac{\xi_4^{(kn)}\left\langle \lambda^{(k)}, \lambda^{(n)}\right\rangle_4 - \lambda_4^{(n)}\left\langle \lambda^{(k)}, \xi^{(kn)}\right\rangle_4}{\left\langle \lambda^{(n)}, \xi^{(kn)}\right\rangle_4^3}\left(1 + \left\langle \frac{d\lambda^{(n)}}{ds_n}, \xi^{(kn)}\right\rangle_4\right) + \frac{\frac{d\lambda_4^{(n)}}{ds_n}\left\langle \lambda^{(k)}, \xi^{(kn)}\right\rangle_4 - \xi_4^{(kn)}\left\langle \lambda^{(k)}, \frac{d\lambda^{(n)}}{ds_n}\right\rangle_4}{\left\langle \lambda^{(n)}, \xi^{(kn)}\right\rangle_4^2}\right\} + F_{rad}^{(k)}, (k=1,2)$$

and the 4-th, 8-th and 12-th equations for the 3-body problem:

$$\frac{d\lambda_4^{(k)}}{ds_k} = \sum_{n=1, n\neq k}^{3}\frac{e_k e_n}{m_k c^2}\left\{\frac{\xi_4^{(kn)}\left\langle \lambda^{(k)}, \lambda^{(n)}\right\rangle_4 - \lambda_4^{(n)}\left\langle \lambda^{(k)}, \xi^{(kn)}\right\rangle_4}{\left\langle \lambda^{(n)}, \xi^{(kn)}\right\rangle_4^3}\left(1 + \left\langle \frac{d\lambda^{(n)}}{ds_n}, \xi^{(kn)}\right\rangle_4\right) + \frac{\frac{d\lambda_4^{(n)}}{ds_n}\left\langle \lambda^{(k)}, \xi^{(kn)}\right\rangle_4 - \xi_4^{(kn)}\left\langle \lambda^{(k)}, \frac{d\lambda^{(n)}}{ds_n}\right\rangle_4}{\left\langle \lambda^{(n)}, \xi^{(kn)}\right\rangle_4^2}\right\} + F_{rad}^{(k)}$$

Following [3] to pass to Euclidian coordinates we consider the relations from [3]:

$$\frac{d\lambda_4^{(k)}}{ds_k} = \frac{ic}{\Delta_k}\frac{d}{dt}\left(\frac{1}{\Delta_k}\right) = \frac{ic\left\langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(k)}(t)\right\rangle}{\Delta_k^4}\,;\; \frac{d\lambda_4^{(n)}}{ds_n} = \frac{icD_{kn}\left\langle \vec{u}^{(n)}(t-\tau_{kn}), \dot{\vec{u}}^{(n)}(t-\tau_{kn})\right\rangle}{\Delta_{kn}^4}\,;$$

$$\left\langle \lambda^{(k)}, \lambda^{(n)}\right\rangle_4 = \frac{\left\langle \vec{u}^{(k)}(t), \vec{u}^{(n)}(t-\tau_{kn})\right\rangle - c^2}{\Delta_k \Delta_{kn}}\,;\; \left\langle \lambda^{(k)}, \xi^{(kn)}\right\rangle_4 = \frac{\left\langle \vec{u}^{(k)}(t), \vec{\xi}^{(kn)}\right\rangle - c^2\tau_{kn}}{\Delta_k}\,;$$

$$\left\langle \lambda^{(n)}, \xi^{(kn)}\right\rangle_4 = \frac{\left\langle \vec{u}^{(n)}(t-\tau_{kn}), \vec{\xi}^{(kn)}\right\rangle - c^2\tau_{kn}}{\Delta_{kn}}\,;$$

$$\left\langle \xi^{(kn)}, \frac{d\lambda^{(n)}}{ds_n}\right\rangle_4 = D_{kn}\left(\frac{\left\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)}(t-\tau_{kn})\right\rangle}{\Delta_{kn}^2} + \frac{\left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)}(t-\tau_{kn})\right\rangle - c^2\tau_{kn}}{\Delta_{kn}^4}\left\langle \vec{u}^{(n)}(t-\tau_{kn}), \dot{\vec{u}}^{(n)}(t-\tau_{kn})\right\rangle\right);$$

;

$$\left\langle \lambda^{(k)}, \frac{d\lambda^{(n)}}{ds_n} \right\rangle_4 = \frac{D_{kn}}{\Delta_k} \left(\frac{\left\langle \vec{u}^{(k)}(t), \dot{\vec{u}}^{(n)}(t-\tau_{kn}) \right\rangle}{\Delta_{kn}^2} + \frac{\left\langle \vec{u}^{(k)}(t), \vec{u}^{(n)}(t-\tau_{kn}) \right\rangle - c^2\tau_{kn}}{\Delta_{kn}^4} \left\langle \vec{u}^{(n)}(t-\tau_{kn}), \dot{\vec{u}}^{(n)}(t-\tau_{kn}) \right\rangle \right);$$

$$H_{kn} = 1 + D_{kn} \left(\frac{\left\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \right\rangle}{\Delta_{kn}^2} + \frac{\left(\left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \right\rangle - c^2\tau_{kn} \right) \left\langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle}{\Delta_{kn}^4} \right).$$

The Dirac 's assumption (cf. [11]) $\tau_k^{ret} = \tau_k^{adv} = \tau$ ($\tau \approx 10^{-24}$ sec) implies $R_{rad}^{(k)} = -\frac{ce_k^2}{\Delta_k^4} \left\langle \vec{u}^{(k)}, \ddot{\vec{u}}^{(k)} \right\rangle$.

By $\beta = \bar{c}/c = 1/137$ we denote the Sommerfeld fine structure constant (cf. [8]).
For 2-body problem we have two energy equations :

$$\frac{m_k \left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \right\rangle}{\Delta_k^3} =$$

$$= \frac{e_k e_n}{c^2} \left\{ \frac{\left\langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \right\rangle - \tau_{kn} \left\langle \vec{u}^{(k)}, \vec{u}^{(n)} \right\rangle}{\left(\left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \right\rangle - c^2\tau_{kn} \right)^3} \Delta_{kn}{}^2 H_{kn} + \frac{\left(\left\langle \xi^{(kn)}, \vec{u}^{(k)} \right\rangle - c^2\tau_{kn} \right) D_{kn} \left\langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle}{\Delta_{kn}^2 \left(c^2\tau_{kn} - \left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \right\rangle \right)^2} - \tau_{kn} D_{kn} \frac{\Delta_{\tau_{kn}}^2 \left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(n)} \right\rangle + \left(\left\langle \vec{u}^{(k)}, \vec{u}^{(n)} \right\rangle - c^2 \right) \left\langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle}{\Delta_{\tau_{kn}}^2 \left(c^2\tau_{kn} - \left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \right\rangle \right)^2} \right\} -$$

$$- \frac{ce_k^2}{\Delta_k^4} \left\langle \vec{u}^{(k)}, \ddot{\vec{u}}^{(k)} \right\rangle$$

while for 3-body problem – three energy equations (cf. [6]):

$$\frac{m_k \left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \right\rangle}{\Delta_k^3} =$$

$$= \sum_{n=1, n\neq k}^{3} \frac{e_k e_n}{c^2} \left\{ \frac{\left\langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \right\rangle - \tau_{kn} \left\langle \vec{u}^{(k)}, \vec{u}^{(n)} \right\rangle}{\left(\left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \right\rangle - c^2\tau_{kn} \right)^3} \Delta_{kn}{}^2 H_{kn} + \frac{\left(\left\langle \xi^{(kn)}, \vec{u}^{(k)} \right\rangle - c^2\tau_{kn} \right) D_{kn} \left\langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle}{\Delta_{kn}^2 \left(c^2\tau_{kn} - \left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \right\rangle \right)^2} - \tau_{kn} D_{kn} \frac{\Delta_{\tau_{kn}}^2 \left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(n)} \right\rangle + \left(\left\langle \vec{u}^{(k)}, \vec{u}^{(n)} \right\rangle - c^2 \right) \left\langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \right\rangle}{\Delta_{\tau_{kn}}^2 \left(c^2\tau_{kn} - \left\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \right\rangle \right)^2} \right\} -$$

$$- \frac{ce_k^2}{\Delta_k^4} \left\langle \vec{u}^{(k)}, \ddot{\vec{u}}^{(k)} \right\rangle.$$

The Kepler Form of the Energy Equations for 2- and 3-Body Problems

In view of the formulas for energy (cf. [7])

$$E_k = m_k c^2 / \sqrt{1 - \frac{\left\langle \vec{u}^{(k)}, \vec{u}^{(k)} \right\rangle}{c^2}} = \frac{m_k c^3}{\Delta_k}, \frac{dE_k}{dt} = -\frac{1}{2} \frac{-2\left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \right\rangle m_k c^3}{\Delta_k{}^3} = \frac{m_k c^3 \left\langle \vec{u}^{(k)}, \dot{\vec{u}}^{(k)} \right\rangle}{\Delta_k{}^3}.$$

we can write down the energy equations in detail for the 2-body problem

$$\frac{m_1 c^3 \left\langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \right\rangle}{\Delta_1^3} = -\frac{ce_1^2}{\Delta_1{}^4} \left\langle \vec{u}^{(1)}, \ddot{\vec{u}}^{(1)} \right\rangle +$$

$$= ce_1 e_2 \left\{ \frac{\left\langle \vec{\xi}^{(12)}, \vec{u}^{(1)} \right\rangle - \tau_{12} \left\langle \vec{u}^{(1)}, \vec{u}^{(2)} \right\rangle}{\left(\left\langle \vec{\xi}^{(12)}, \vec{u}^{(2)} \right\rangle - c^2\tau_{12} \right)^3} \Delta_{12}^2 H_{12} + \frac{\left(\left\langle \vec{\xi}^{(12)}, \vec{u}^{(1)} \right\rangle - c^2\tau_{12} \right) D_{12} \left\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \right\rangle}{\Delta_{12}^2 \left(c^2\tau_{12} - \left\langle \vec{\xi}^{(12)}, \vec{u}^{(2)} \right\rangle \right)^2} - \tau_{12} D_{12} \frac{\Delta_{12}^2 \left\langle \vec{u}^{(1)}, \dot{\vec{u}}^{(2)} \right\rangle + \left(\left\langle \vec{u}^{(1)}, \vec{u}^{(2)} \right\rangle - c^2 \right) \left\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \right\rangle}{\Delta_{12}^2 \left(c^2\tau_{12} - \left\langle \vec{\xi}^{(12)}, \vec{u}^{(2)} \right\rangle \right)^2} \right\},$$

$$\frac{m_2 c^3 \left\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \right\rangle}{\Delta_2^3} = -\frac{ce_2^2}{\Delta_2{}^4} \left\langle \vec{u}^{(2)}, \ddot{\vec{u}}^{(2)} \right\rangle +$$

$$= ce_1 e_2 \left\{ \frac{\left\langle \vec{\xi}^{(21)}, \vec{u}^{(2)} \right\rangle - \tau_{21} \left\langle \vec{u}^{(2)}, \vec{u}^{(1)} \right\rangle}{\left(\left\langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \right\rangle - c^2\tau_{21} \right)^3} \Delta_{21}^2 H_{21} + \frac{\left(\left\langle \vec{\xi}^{(21)}, \vec{u}^{(2)} \right\rangle - c^2\tau_{21} \right) D_{21} \left\langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \right\rangle}{\Delta_{21}^2 \left(c^2\tau_{21} - \left\langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \right\rangle \right)^2} - \tau_{21} D_{21} \frac{\Delta_{21}^2 \left\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(1)} \right\rangle + \left(\left\langle \vec{u}^{(2)}, \vec{u}^{(1)} \right\rangle - c^2 \right) \left\langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \right\rangle}{\Delta_{21}^2 \left(c^2\tau_{21} - \left\langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \right\rangle \right)^2} \right\},$$

and for 3-body problem

$$\frac{m_1 c^3 \left\langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \right\rangle}{\Delta_1^3} = -\frac{ce_1^2}{\Delta_1{}^4} \left\langle \vec{u}^{(1)}, \ddot{\vec{u}}^{(1)} \right\rangle +$$

$$= ce_1 e_2 \left\{ \frac{\left\langle \vec{\xi}^{(12)}, \vec{u}^{(1)} \right\rangle - \tau_{12} \left\langle \vec{u}^{(1)}, \vec{u}^{(2)} \right\rangle}{\left(\left\langle \vec{\xi}^{(12)}, \vec{u}^{(2)} \right\rangle - c^2\tau_{12} \right)^3} \Delta_{12}^2 H_{12} + \frac{\left(\left\langle \vec{\xi}^{(12)}, \vec{u}^{(1)} \right\rangle - c^2\tau_{12} \right) D_{12} \left\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \right\rangle}{\Delta_{12}^2 \left(c^2\tau_{12} - \left\langle \vec{\xi}^{(12)}, \vec{u}^{(2)} \right\rangle \right)^2} - \tau_{12} D_{12} \frac{\Delta_{12}^2 \left\langle \vec{u}^{(1)}, \dot{\vec{u}}^{(2)} \right\rangle + \left(\left\langle \vec{u}^{(1)}, \vec{u}^{(2)} \right\rangle - c^2 \right) \left\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \right\rangle}{\Delta_{12}^2 \left(c^2\tau_{12} - \left\langle \vec{\xi}^{(12)}, \vec{u}^{(2)} \right\rangle \right)^2} \right\} +$$

$$+ ce_1 e_3 \left\{ \frac{\left\langle \vec{\xi}^{(13)}, \vec{u}^{(1)} \right\rangle - \tau_{13} \left\langle \vec{u}^{(1)}, \vec{u}^{(3)} \right\rangle}{\left(\left\langle \vec{\xi}^{(13)}, \vec{u}^{(3)} \right\rangle - c^2\tau_{13} \right)^3} \Delta_{13}^2 H_{13} + \frac{\left(\left\langle \vec{\xi}^{(13)}, \vec{u}^{(1)} \right\rangle - c^2\tau_{13} \right) D_{13} \left\langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \right\rangle}{\Delta_{13}^2 \left(c^2\tau_{13} - \left\langle \vec{\xi}^{(13)}, \vec{u}^{(3)} \right\rangle \right)^2} - \tau_{13} D_{13} \frac{\Delta_{13}^2 \left\langle \vec{u}^{(1)}, \dot{\vec{u}}^{(3)} \right\rangle + \left(\left\langle \vec{u}^{(1)}, \vec{u}^{(3)} \right\rangle - c^2 \right) \left\langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \right\rangle}{\Delta_{13}^2 \left(c^2\tau_{13} - \left\langle \vec{\xi}^{(13)}, \vec{u}^{(3)} \right\rangle \right)^2} \right\};$$

$$\frac{m_2 c^3 \left\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \right\rangle}{\Delta_2^3} = -\frac{ce_2^2}{\Delta_2{}^4} \left\langle \vec{u}^{(2)}, \ddot{\vec{u}}^{(2)} \right\rangle +$$

$$= ce_2 e_1 \left\{ \frac{\langle \vec{\xi}^{(21)}, \vec{u}^{(2)} \rangle - \tau_{21} \langle \vec{u}^{(2)}, \vec{u}^{(1)} \rangle}{\left(\langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \rangle - c^2 \tau_{21}\right)^3} \Delta_{21}^2 H_{21} + \frac{\left(\langle \vec{\xi}^{(21)}, \vec{u}^{(2)} \rangle - c^2 \tau_{21}\right) D_{21} \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{21}^2 \left(c^2 \tau_{21} - \langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \rangle\right)^2} - \tau_{21} D_{21} \frac{\Delta_{21}^2 \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(1)} \rangle + \left(\langle \vec{u}^{(2)}, \vec{u}^{(1)} \rangle - c^2\right) \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{21}^2 \left(c^2 \tau_{21} - \langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \rangle\right)^2} \right\} +$$

$$+ ce_2 e_3 \left\{ \frac{\langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \rangle - \tau_{23} \langle \vec{u}^{(k)}, \vec{u}^{(n)} \rangle}{\left(\langle \vec{\xi}^{(23)}, \vec{u}^{(n)} \rangle - c^2 \tau_{23}\right)^3} \Delta_{23}^2 H_{23} + \frac{\left(\langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \rangle - c^2 \tau_{23}\right) D_{23} \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle}{\Delta_{23}^2 \left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle\right)^2} - \tau_{23} D_{23} \frac{\Delta_{23}^2 \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(3)} \rangle + \left(\langle \vec{u}^{(2)}, \vec{u}^{(3)} \rangle - c^2\right) \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle}{\Delta_{23}^2 \left(c^2 \tau_{23} - \langle \vec{\xi}^{(23)}, \vec{u}^{(3)} \rangle\right)^2} \right\};$$

$$\frac{m_3 c^3 \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle}{\Delta_3^3} = -\frac{ce_3^2}{\Delta_3^4} \langle \vec{u}^{(3)}, \ddot{\vec{u}}^{(3)} \rangle +$$

$$+ ce_3 e_1 \left\{ \frac{\langle \vec{\xi}^{(31)}, \vec{u}^{(3)} \rangle - \tau_{31} \langle \vec{u}^{(3)}, \vec{u}^{(1)} \rangle}{\left(\langle \vec{\xi}^{(31)}, \vec{u}^{(1)} \rangle - c^2 \tau_{31}\right)^3} \Delta_{31}^2 H_{31} + \frac{\left(\langle \vec{\xi}^{(31)}, \vec{u}^{(3)} \rangle - c^2 \tau_{31}\right) D_{31} \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{31}^2 \left(c^2 \tau_{31} - \langle \vec{\xi}^{(31)}, \vec{u}^{(1)} \rangle\right)^2} - \tau_{31} D_{31} \frac{\Delta_{31}^2 \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(1)} \rangle + \left(\langle \vec{u}^{(3)}, \vec{u}^{(1)} \rangle - c^2\right) \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{31}^2 \left(c^2 \tau_{31} - \langle \vec{\xi}^{(31)}, \vec{u}^{(1)} \rangle\right)^2} \right\} +$$

$$+ ce_3 e_2 \left\{ \frac{\langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle - \tau_{32} \langle \vec{u}^{(3)}, \vec{u}^{(2)} \rangle}{\left(\langle \vec{\xi}^{(32)}, \vec{u}^{(2)} \rangle - c^2 \tau_{32}\right)^3} \Delta_{32}^2 H_{32} + \frac{\left(\langle \vec{\xi}^{(32)}, \vec{u}^{(k)} \rangle - c^2 \tau_{32}\right) D_{32} \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \rangle}{\Delta_{32}^2 \left(c^2 \tau_{32} - \langle \vec{\xi}^{(32)}, \vec{u}^{(2)} \rangle\right)^2} - \tau_{32} D_{32} \frac{\Delta_{32}^2 \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(2)} \rangle + \left(\langle \vec{u}^{(3)}, \vec{u}^{(2)} \rangle - c^2\right) \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \rangle}{\Delta_{32}^2 \left(c^2 \tau_{32} - \langle \vec{\xi}^{(32)}, \vec{u}^{(2)} \rangle\right)^2} \right\}.$$

In the 3D-Kepler formulation the first particle P_1 is put at the origin O (0,0,0), that is,

$$P_1\left(x_1^{(1)}(t) = 0, x_2^{(1)}(t) = 0, x_3^{(1)}(t) = 0\right),\ t \in [0,\infty) \Rightarrow E_{kin}^{(1)}(t) = 0. \qquad (1)$$

From [4] since $\beta = \bar{c}/c = 1/137$ and $\langle \vec{u}(t-\tau_{kn}), \vec{u}(t-\tau_{kn}) \rangle \le \bar{c}^2 \ll c^2; \Delta_{kn} = \sqrt{c^2 - \langle \vec{u}(t-\tau_{kn}), \vec{u}(t-\tau_{kn}) \rangle} \approx c$,

we get $c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle \approx c^2 \tau_{kn}$; $D_{kn} = \dfrac{c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle}{c^2 \tau_{kn} - \langle \vec{\xi}^{(kn)}, \vec{u}^{(k)} \rangle} \approx 1; (k = 1,2,3), n \ne k$ and

$$H_{21} = 1 + D_{21} \left(\frac{\langle \vec{\xi}^{(21)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{21}^2} + \frac{\left(\langle \vec{\xi}^{(21)}, \vec{u}^{(1)} \rangle - c^2 \tau_{21}\right) \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{21}^4} \right) \approx 1 + \frac{\langle \vec{\xi}^{(21)}, \dot{\vec{u}}^{(1)} \rangle - \tau_{21} \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{c^2} = 1;$$

$$H_{31} = 1 + D_{31} \left(\frac{\langle \vec{\xi}^{(31)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{31}^2} + \frac{\left(\langle \vec{\xi}^{(31)}, \vec{u}^{(1)} \rangle - c^2 \tau_{31}\right) \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{\Delta_{31}^4} \right) \approx 1 + \frac{\langle \vec{\xi}^{(31)}, \dot{\vec{u}}^{(1)} \rangle - \tau_{31} \langle \vec{u}^{(1)}, \dot{\vec{u}}^{(1)} \rangle}{c^2} = 1;$$

$$H_{kn} = 1 + D_{kn} \left(\frac{\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \rangle}{\Delta_{kn}^2} + \frac{\left(\langle \vec{\xi}^{(kn)}, \vec{u}^{(n)} \rangle - c^2 \tau_{kn}\right) \langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \rangle}{\Delta_{kn}^4} \right) \approx 1 + \frac{\langle \vec{\xi}^{(kn)}, \dot{\vec{u}}^{(n)} \rangle - \tau_{kn} \langle \vec{u}^{(n)}, \dot{\vec{u}}^{(n)} \rangle}{c^2} \le$$

$$\le 1 + \frac{\tau_{kn} \left\| \dot{\vec{u}}^{(n)} \right\| (1+\beta)}{c} \le 1 + \frac{\tau_{kn} \bar{c} \omega (1+\beta)}{c} = 1 + \tau_{kn} \omega \beta (1+\beta) \approx 1 + \tau_{kn} \omega \beta.$$

Assume that $e_1 = +e_{el}, e_2 = -e_{el}; e_3 = -e_{el}; e_{el} = -1{,}6 \times 10^{-19} C$. Then in view of (1) we get just one equation for the moving particle (electron):

$$0 = 0; \quad \frac{dE_2}{dt} = e_{el}^2 \left(\frac{\langle \vec{\xi}^{(21)}, \vec{u}^{(2)} \rangle}{c^3 \tau_{21}^3} - \frac{\langle \vec{u}^{(2)}, \ddot{\vec{u}}^{(2)} \rangle}{c^3} \right) \equiv \frac{dE}{dt} = e_{el}^2 \left(\frac{\langle \vec{\xi}^{(21)}, \vec{u} \rangle}{c^3 \tau_{21}^3} - \frac{\langle \vec{u}, \ddot{\vec{u}} \rangle}{c^3} \right) \qquad (2)$$

and two equations for 3-body problem:

$0 = 0$;

$$\frac{dE_2}{dt} = e_{el}^2 \frac{\langle \vec{\xi}^{(21)}, \vec{u}^{(2)} \rangle}{c^3 \tau_{21}^3} + e_{el}^2 \left[\frac{\tau_{23} \langle \vec{u}^{(2)}, \vec{u}^{(3)} \rangle - \langle \vec{\xi}^{(23)}, \vec{u}^{(2)} \rangle}{c^3 \tau_{23}^3} \left(1 + \frac{\langle \vec{\xi}^{(23)}, \dot{\vec{u}}^{(3)} \rangle - \tau_{23} \langle \vec{u}^{(3)}, \dot{\vec{u}}^{(3)} \rangle}{c^2} \right) - \frac{\langle \vec{u}^{(2)}, \dot{\vec{u}}^{(3)} \rangle}{c^3 \tau_{23}} \right] - \frac{ce_{el}^2}{\Delta_2^4} \langle \vec{u}^{(2)}, \ddot{\vec{u}}^{(2)} \rangle; \quad (3)$$

$$\frac{dE_3}{dt} = e_{el}^2 \frac{\langle \vec{\xi}^{(31)}, \vec{u}^{(3)} \rangle}{c^3 \tau_{31}^3} + e_{el}^2 \left[\frac{\tau_{32} \langle \vec{u}^{(3)}, \vec{u}^{(2)} \rangle - \langle \vec{\xi}^{(32)}, \vec{u}^{(3)} \rangle}{c^3 \tau_{32}^3} \left(1 + \frac{\langle \vec{\xi}^{(32)}, \dot{\vec{u}}^{(2)} \rangle - \tau_{32} \langle \vec{u}^{(2)}, \dot{\vec{u}}^{(2)} \rangle}{c^2} \right) - \frac{\langle \vec{u}^{(3)}, \dot{\vec{u}}^{(2)} \rangle}{c^3 \tau_{32}} \right] - \frac{ce_{el}^2}{\Delta_3^4} \langle \vec{u}^{(3)}, \ddot{\vec{u}}^{(3)} \rangle.$$

Remark 1. We would like to point out that all functions $\vec{x}^{(k)}, \vec{u}^{(k)}, \vec{\xi}^{(kn)} = \vec{x}^{(k)}(t) - \vec{x}^{(n)}(t - \tau_{kn}), \tau_{kn} = \frac{1}{c}\sqrt{\langle \vec{\xi}^{(kn)}, \vec{\xi}^{(kn)} \rangle}$ in the right-hand sides of (2) and (3) are known as solutions of the first three equations of motion. The solution belongs to spaces introduced in [6].

The Energy Equations in Spherical Coordinates

We pass to the spherical coordinates, that is, the particles $P_1 = (0,0,0)$, $P_k (k = 2,3)$, are located at the point

$x_1^{(k)}(t) = \rho_k(t) \cos\varphi_k(t) \cos\lambda_k(t)$; $x_2^{(k)}(t) = \rho_k(t) \sin\varphi_k(t) \cos\lambda_k(t)$; $x_3^{(k)}(t) = \rho_k(t) \sin\lambda_k(t)$, (cf. [6]),

where $\rho_k \ge 0; \varphi_k \ge 0; \lambda_k \in \left[-\frac{\pi}{2} + \delta, \frac{\pi}{2} - \delta\right], 0 < \delta < \frac{\pi}{2}$ and then the velocities are:

$$u_1^{(k)} = \dot{\rho}_k \cos\varphi_k \cos\lambda_k - \rho_k \dot{\varphi}_k \sin\varphi_k \cos\lambda_k - \rho_k \dot{\lambda}_k \cos\varphi_k \sin\lambda_k$$

$$u_2^{(k)} = \dot\rho_k \sin\varphi_k \cos\lambda_k + \rho_k \dot\varphi_k \cos\varphi_k \cos\lambda_k - \rho_k \dot\lambda_k \sin\varphi_k \sin\lambda_k$$
$$u_3^{(k)} = \dot\rho_k \sin\lambda_k + \rho_k \dot\lambda_k \cos\lambda_k$$

where $\dot\rho^{(k)} = d\rho^{(k)}/dt$. For the accelerations we have:

$$\dot u_1^{(k)} \approx \ddot\rho_k \cos\varphi_k \cos\lambda_k - \rho_k \ddot\varphi_k \sin\varphi_k \cos\lambda_k - \rho_k \ddot\lambda_k \cos\varphi_k \sin\lambda_k$$
$$\dot u_2^{(k)} \approx \ddot\rho_k \sin\varphi_k \cos\lambda_k + \rho_k \ddot\varphi_k \cos\varphi_k \cos\lambda_k - \rho_k \ddot\lambda_k \sin\varphi_k \sin\lambda_k$$
$$\dot u_3^{(k)} \approx \ddot\rho_k \sin\lambda_k + \rho_k \ddot\lambda_k \cos\lambda_k \,.$$

Second derivatives we approximate by the formulas $(k = 2, 3)$:

$$\ddot u_1^{(k)} \approx \dddot\rho_k \cos\varphi_k \cos\lambda_k - \dddot\varphi_k \rho_k \sin\varphi_k \cos\lambda_k - \dddot\lambda_k \rho_k \cos\varphi_k \sin\lambda_k$$
$$\ddot u_2^{(k)} \approx \dddot\rho_k \sin\varphi_k \cos\lambda_k + \dddot\varphi_k \rho_k \cos\varphi_k \cos\lambda_k - \dddot\lambda_k \rho_k \sin\varphi_k \sin\lambda_k$$
$$\ddot u_3^{(k)} \approx \dddot\rho_k \sin\lambda_k + \dddot\lambda_k \rho_k \cos\lambda_k .$$

Assume that $\omega_2 = \omega_3 = \omega$. Then $\sqrt{\langle \vec u^{(k)}, \vec u^{(k)} \rangle} = \sqrt{e^{2\mu T} R_k^{\,2} + e^{2\mu T} \rho_k^{\,2} \Phi_k^{\,2} + e^{2\mu T} \rho_k^{\,2} Y_k^2} \le \bar c$.

After substitution $r_k = \dot\rho_k, \phi_k = \dot\varphi_k, \eta_k = \dot\lambda_k$ we obtain

$$\left\|\dot{\vec u}^{(n)}\right\| = \sqrt{\langle \dot{\vec u}^{(n)}, \dot{\vec u}^{(n)} \rangle} = \sqrt{\left|\dot r_n^{\,2} + \rho_n^{\,2} \dot\phi_n^{\,2} \cos^2\lambda_n + \rho_n^{\,2} \dot\eta_n^{\,2}\right|} \le \omega \sqrt{e^{2\mu T} R_n^{\,2} + e^{2\mu T} \rho_n^{\,2} \Phi_n^{\,2} + e^{2\mu T} \rho_n^{\,2} Y_n^2} \le \omega \bar c \ \ (n = 2, 3);$$
$$\left\|\ddot{\vec u}^{(n)}\right\| = \sqrt{\langle \ddot{\vec u}^{(n)}, \ddot{\vec u}^{(n)} \rangle} = \sqrt{\left|\ddot r_n^{\,2} + \rho_n^{\,2} \ddot\phi_n^{\,2} \cos^2\lambda_n + \rho_n^{\,2} \ddot\eta_n^{\,2}\right|} \le \omega^2 \sqrt{e^{2\mu T} R_n^{\,2} + e^{2\mu T} \rho_n^{\,2} \Phi_n^{\,2} + e^{2\mu T} \rho_n^{\,2} Y_n^2} \le \omega^2 \bar c \,.$$

For the second particle we have $\rho_2 = \rho_2(t)$; $\varphi_2 = \varphi_2(t)$; $\rho_3 = \rho_3(t - \tau_{23})$; $\varphi_3 = \varphi_3(t - \tau_{23})$;

$$\tau_{21}(t) = \frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(2)}(t) - x_\alpha^{(1)}(t - \tau_{21}(t))\right]^2} = \frac{\rho_2(t)}{c};\ \tau_{31}(t) = \frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(3)}(t) - x_\alpha^{(1)}(t - \tau_{21}(t))\right]^2} = \frac{\rho_3(t)}{c};$$

$$\rho_{21}(t) = \rho_2(t) = \rho_{20} + \int_0^t r_2(s)\,ds \ge \rho_{20} - R_2 \frac{e^{\mu T} - 1}{\mu} \approx \rho_{20};\ \rho_{31}(t) = \rho_3(t) = \rho_{30} + \int_0^t r_3(s)\,ds \ge \rho_{30} - R_3 \frac{e^{\mu T} - 1}{\mu} \approx \rho_{30};$$

$$\vec\xi^{(23)} = (0,\ \rho_2 \cos\varphi_2 - \rho_3 \cos\varphi_3,\ \rho_2 \sin\varphi_2 - \rho_3 \sin\varphi_3);$$
$$\tau_{23}(t) = \frac{1}{c}\sqrt{\langle \vec\xi^{(23)}, \vec\xi^{(23)} \rangle} \equiv \frac{1}{c}\sqrt{\sum_{\alpha=1}^{3}\left[x_\alpha^{(2)}(t) - x_\alpha^{(3)}(t - \tau_{23}(t))\right]^2} = \frac{1}{c}\sqrt{\rho_2^2 + \rho_3^2 - 2\rho_2\rho_3 \cos(\varphi_2 - \varphi_3)} \ge \frac{\left|\rho_2(t) - \rho_3(t - \tau_{23})\right|}{c};$$

$$\left|\rho_2(t) - \rho_3(t - \tau_{23})\right| = \left|\rho_{20} + \int_0^t r_2(s)\,ds - \rho_{30} - \int_0^{t-\tau_{23}} r_3(s)\,ds\right| = \left|\rho_{20} + \int_0^t r_2(s)\,ds - \rho_{30} - \int_0^t r_3(s)\,ds - \int_{t-\tau_{23}}^{t} r_3(s)\,ds\right| \ge$$
$$\ge \left|\rho_{20} - \rho_{30}\right| - R_3 \frac{e^{\mu t} - e^{\mu(t - \tau_{23})}}{\mu} = \left|\rho_{20} - \rho_{30}\right| - R_3 e^{\mu t} \frac{1 - e^{-\mu\tau_{23}}}{\mu} \ge \left|\rho_{20} - \rho_{30}\right| - \frac{\bar c}{\mu} \approx \left|\rho_{20} - \rho_{30}\right|;$$
$$\left|\rho_2(t) - \rho_3(t)\right| \ge \left|\rho_{20} - \rho_{30}\right| - \frac{R_2 + R_3}{\mu} e^{\mu T} > \left|\rho_{20} - \rho_{30}\right| - \frac{2\bar c}{\mu} > \left|\rho_{20} - \rho_{30}\right|;\ \ \frac{1}{\tau_{23}(t)} \le \frac{c}{\left|\rho_{20} - \rho_{30}\right|} .$$

Then in view of $\beta^2 = 1/137^2 \ \square\ 0$ and $T\omega = 2\pi$ one obtains for the 2-body problem

$$\left|E(t)\right| = e_{el}^{\,2}\left|\int_0^T \left(\frac{\langle \vec\xi^{(21)}, \vec u \rangle}{c^3 \tau_{21}^{\,3}} - \frac{\langle \vec u, \ddot{\vec u} \rangle}{c^3}\right) ds\right| \le e_{el}^{\,2}\int_0^T \left(\frac{\sqrt{\langle \vec\xi^{(21)}, \vec\xi^{(21)} \rangle}\sqrt{\langle \vec u, \vec u \rangle}}{c^3 \tau_{21}^{\,3}} + \frac{\sqrt{\langle \vec u, \vec u \rangle}\sqrt{\langle \ddot{\vec u}, \ddot{\vec u} \rangle}}{c^3}\right) ds \le$$
$$\le e_{el}^{\,2}\int_0^T \left(\frac{c\tau_{21}\bar c}{c^3 \tau_{21}^{\,3}} + \frac{\bar c^2 \omega^2}{c^3}\right) e^{\mu s} ds \le e_{el}^{\,2}\int_0^T \left(\frac{\bar c}{c^2 \tau_{21}^{\,2}} + \frac{\bar c^2 \omega^2}{c^3}\right) e^{\mu s} ds = e_{el}^{\,2}\left(\frac{c\beta}{\rho_2^{\,2}} + \frac{\beta^2 \omega^2}{c}\right)\frac{e^{\mu T} - 1}{\mu} = e_{el}^{\,2}\beta\left(\frac{cT}{\rho_2^{\,2}} + \frac{\beta T^2 \omega^2}{cT}\right)\frac{e^{\mu T} - 1}{\mu T} = \qquad (4)$$
$$= e_{el}^{\,2}\beta\left(\frac{cT}{\rho_2^{\,2}} + \frac{4\pi^2 \beta}{cT}\right)\frac{e^{\mu T} - 1}{\mu T}.$$

For the second particle of the 3-body problem we get

$$\left|E_2(t)\right| \le \int_0^t \left[e_2^{\,2}\frac{\omega^2 \bar c^2}{c^3} + \left|e_2 e_1\right|\frac{c\bar c}{c^3 \tau_{21}^{\,2}} + \left|e_2 e_3\right|\left(\frac{c\bar c + \bar c^2}{c^3 \tau_{23}^{\,2}}(1 + \tau_{23}\omega\beta) + \frac{\bar c^2 \omega}{c^3 \tau_{23}}\right)\right] ds \le$$
$$\le \int_0^t \left[e_2^{\,2}\frac{\omega^2 \beta^2}{c} + \left|e_2 e_1\right|\frac{\beta}{c\tau_{21}^{\,2}} + \left|e_2 e_3\right|\left(\frac{\beta + \beta^2}{c\tau_{23}^{\,2}} + \omega\frac{\beta^2 + 2\beta^3 + \beta^4 + \beta^2}{c\tau_{23}}\right)\right] ds \approx$$
$$\approx \left[e_2^{\,2}\frac{\omega^2 \beta^2}{c} + \left|e_2 e_1\right|\frac{\beta}{c}\frac{c^2}{\rho_2^{\,2}} + \left|e_2 e_3\right|\left(\frac{\beta}{c}\frac{c^2}{\left|\rho_{20} - \rho_{30}\right|^2} + \omega\frac{2\beta^2}{c}\frac{c}{\left|\rho_{20} - \rho_{30}\right|}\right)\right]\frac{e^{\mu T} - 1}{\mu} = \qquad (5)$$
$$= e_{el}^{\,2}\left[\frac{\omega^2 T^2 \beta^2}{cT} + \frac{cT\beta}{\rho_2^{\,2}} + \frac{cT\beta}{\left|\rho_{20} - \rho_{30}\right|^2} + \frac{2\beta^2 T\omega}{\left|\rho_{20} - \rho_{30}\right|}\right]\frac{e^{\mu T} - 1}{\mu T} = e_{el}^{\,2}\beta\left[\frac{4\pi^2 \beta}{cT} + \frac{cT}{\rho_2^{\,2}} + \frac{cT}{\left|\rho_{20} - \rho_{30}\right|^2} + \frac{4\pi\beta}{\left|\rho_{20} - \rho_{30}\right|}\right]\frac{e^{\mu T} - 1}{\mu T}.$$

For the third particle we get $\rho_2 = \rho_2(t-\tau_{32})$; $\varphi_2 = \varphi_2(t-\tau_{32})$; $\rho_3 = \rho_3(t)$; $\varphi_3 = \varphi_3(t)$;

$\vec{\xi}^{(32)} = (0,\ \rho_3\cos\varphi_3 - \rho_2\cos\varphi_2,\ \rho_3\sin\varphi_3 - \rho_2\sin\varphi_2)$;

$$\tau_{32}(t) = \frac{1}{c}\sqrt{\langle \vec{\xi}^{(32)}, \vec{\xi}^{(32)} \rangle} = \frac{1}{c}\sqrt{\rho_3^2 + \rho_2^2 - 2\rho_2\rho_3\cos(\varphi_3 - \varphi_2)} \ge \frac{|\rho_3(t) - \rho_2(t-\tau_{32})|}{c} \ge \frac{|\rho_{30} - \rho_{20}|}{c};$$

$$|\rho_3(t) - \rho_2(t-\tau_{32})| = \left|\rho_{30} + \int_0^t r_3(s)\,ds - \rho_{20} - \int_0^{t-\tau_{32}} r_2(s)\,ds\right| = \left|\rho_{30} + \int_0^t r_3(s)\,ds - \rho_{20} - \int_0^t r_2(s)\,ds - \int_{t-\tau_{32}}^t r_2(s)\,ds\right| \ge$$

$$\ge |\rho_{30} - \rho_{20}| - R_2\frac{e^{\mu t} - e^{\mu(t-\tau_{32})}}{\mu} = |\rho_{30} - \rho_{20}| - R_2 e^{\mu t}\frac{1 - e^{-\mu\tau_{32}}}{\mu} \ge |\rho_{30} - \rho_{20}| - \frac{\bar{c}}{\mu} \approx |\rho_{30} - \rho_{20}|;\ \frac{1}{\tau_{32}(t)} \le \frac{c}{|\rho_{30} - \rho_{20}|}.$$

Then

$$|E_3(t)| \le e_{el}^2\beta\left(\frac{4\pi^2\beta}{cT} + \frac{cT}{\rho_3^2} + \frac{cT}{|\rho_{20} - \rho_{30}|^2} + \frac{4\pi\beta}{|\rho_{20} - \rho_{30}|}\right)\frac{e^{\mu T} - 1}{\mu T}.$$

Hydrogen and Hydrogen-like Atoms

The numerical values for physical quantities are taken from the monographs [9], [17].

Recall that $e_1 = 1.6\times10^{-19}C$, $e_2 = e_3 = e_{el} = -1.6\times10^{-19}C$, $\beta = 1/137$, $c \approx 3\times10^8\,m/\sec$, $\rho_2 = 5.29\times10^{-11}$.

Then $|e_2e_3|\beta = \frac{1.6^2\times10^{-38}}{137} \approx 0.0186\times10^{-38} = 1.86\times10^{-40}$ and $cT = 3\times10^8\times1.53\times10^{-16} = 4.59\times10^{-8}$.

Let us calculate the energy of the first stationary state from quantum mechanics (cf. [9]).

Since $E_p = -\frac{1}{\varepsilon_0^2}.\frac{e_2^4 m_2}{8\hbar^2}\frac{1}{p^2}\ (p = 1,2,...)$ for $p = 1$ one obtains

$$|E_1| = \left|\frac{1}{\varepsilon_0^2}.\frac{e_2^4 m_2}{8\hbar^2}\right| = \frac{1}{8.86^2\times10^{-24}}.\frac{(1.6\times10^{-19})^4\times9.1\times10^{-31}}{8\times6.62^2\times10^{-68}} = 2.2\times10^{-18}J\times6.24\times10^{18} \approx 13.6eV.$$

For the Hydrogen-atom we use the energy estimation (4) ($2\pi < \mu T$). Let us recall that the condition $\omega < \mu$ is compulsory for the convergence of the successive approximations with accordance of the fixed-point theorem.

We have:

$$|E(t)| \le \beta e_{el}^2\left(\frac{cT}{\rho_2^2} + \frac{4\pi^2\beta}{cT}\right)\frac{e^{\mu T} - 1}{\mu T} = 1.86\times10^{-40}\left(\frac{4.59\times10^{-8}}{\rho_2^2} + \frac{39.478}{137\times4.59\times10^{-8}}\right)\frac{e^{\mu T} - 1}{\mu T} =$$

$$= 10^{-40}\left(3.05\times10^{13} + 1.167\times10^7\right)\frac{e^{\mu T} - 1}{\mu T} \approx 10^{-40}\times3.05\times10^{13}\frac{e^{\mu T} - 1}{\mu T} = 3.05\times10^{-27}\frac{e^{\mu T} - 1}{\mu T}J = \bar{E},$$

$$\bar{E} = 3.05\times10^{-27}\frac{e^{\mu T} - 1}{\mu T}\times6.24\times10^{18} = 1.9\times10^{-8}\frac{e^{\mu T} - 1}{\mu T}eV.$$

But 13,6 eV (cf. [9]) is the ionization work for transition from neutral atom to positive single charge ion.

Then $1.9\times10^{-8}\frac{e^{\mu T} - 1}{\mu T} \le 13.6$.(cf. [9]). It follows $\frac{e^{\mu T} - 1}{\mu T} = 7.158\times10^8$. For $\mu T = 23.54$ one obtains $\frac{e^{\mu T} - 1}{\mu T} = 7.103\times10^8$.

Since $T = 2\pi/\omega = 1.53\times10^{-16}$ it follows $\mu = \frac{23.54}{1.53\times10^{-16}} \approx 1.54\times10^{17}$.

We give an estimation of the initial value of the radius ρ_{20} of the moving electron. From the inequality we obtain

$$1.86\times10^{-40}\left(\frac{4.59\times10^{-8}}{\rho_2^2} + \frac{39.478}{137\times4.59\times10^{-8}}\right)\times7.103\times10^8\times6.24\times10^{18} \le 13.6eV;$$

$$\frac{4.59\times10^{-8}}{\rho_2^2} \le 16.5\times10^{12} - 6.27\times10^6 \approx 16.5\times10^{12} \Leftrightarrow \rho_{20} \ge \sqrt{\frac{4.59\times10^{-8}}{16.5\times10^{12}}} \approx 0.527\times10^{-10} = 5.27\times10^{-11}m.$$

The conclusion is: the energy of the moving electron is in the interval $0 \le E_2(t) \le 13.6 \Leftrightarrow -13.6eV \le -E_2(t) \le 0$.

For the Hydrogen-like atom He^+ the ionization work for transition from a single-charge ion to a positive 2-charge ion is E_{He^+}=54.1 eV (cf. [9]). Then $|E(t)| \le |2e^+e^-|\beta\left(\frac{cT}{\rho_2^2} + \frac{4\pi^2\beta}{cT}\right)\frac{e^{\mu T} - 1}{\mu T}\times6.24\times10^{18} \le 54.1$. We estimate the radius of the circle of the moving electron

$$2\times1.86\times10^{-40}\left(\frac{4.59\times10^{-8}}{\rho_2^{\ 2}}+\frac{39.478}{137\times4.59\times10^{-8}}\right)\times7.103\times10^{8}\times6.24\times10^{18}\le54.1eV;$$

$$\frac{4.59\times10^{-8}}{\rho_2^{\ 2}}\le3.28\times10^{13}-6.27\times10^{6}\Rightarrow\rho_2\ge\sqrt{\frac{45.9}{3.28}}\times10^{-11}=3.74\times10^{-11}.$$

This means that the attraction is stronger than attraction in the Hydrogen atom.
For the Hydrogen-like atom Li^{++} the ionization work for transition from a 2-charge ion to a 3-charge ion is $E_{Li++}=122eV$ (cf. [9]):

$$|E(t)|\le|3e^+e^-|\beta\left(\frac{cT}{\rho_2^{\ 2}}+\frac{4\pi^2\beta}{cT}\right)\frac{e^{\mu T}-1}{\mu T}\times6.24\times10^{18}\le122\,eV;$$

$$\frac{4.59\times10^{-8}}{\rho_2^{\ 2}}\le4.932\times10^{13}-6.27\times10^{6}\Leftrightarrow\rho_2\ge\sqrt{\frac{45.9}{4.932}\times10^{-22}}=3.05\times10^{-11}.$$

For the Hydrogen-like atom Be^{+++} the ionization work for transition from a 3-charge ion to a 4-charge ion is

$E_{Be^{+++}}=217eV$ (cf. [9]): $|E_2(t)|\le|4e^+e^-|\beta\left(\frac{cT}{\rho_2^{\ 2}}+\frac{4\pi^2\beta}{cT}\right)\frac{e^{\mu T}-1}{\mu T}\times6.24\times10^{18}\le217$;

$$\frac{4\times1.86}{10^{40}}\times\left(\frac{4.59\times10^{-8}}{\rho_2^{\ 2}}+\frac{39.478}{137\times4.59\times10^{-8}}\right)\times7.103\times10^{8}\times6.24\times10^{18}\le217;$$

$$\frac{4.59}{\rho_2^{\ 2}}\times10^{-8}\le6.58\times10^{13}-6.27\times10^{6}\approx\rho_2\ge\sqrt{\frac{4.59}{6.58}\times10^{-21}}=2.641\times10^{-11}.$$

Let us consider the Uranian-atom with Z=92. Then the ionization work for the last electron is $W_J=13.6\times Z^2=13.6\times92^2=1{,}56\times10^{5}eV$ (cf. [9]).

Therefore,

$$|E(t)|=\frac{91|e_2e_1|}{137}\left(\frac{cT}{\rho_2^{\ 2}}+\frac{4\pi^2\beta}{cT}\right)\frac{e^{\mu T}-1}{\mu T}\times6.24\times10^{18}=1.7\times10^{-38}\left(\frac{4.59\times10^{-8}}{5.29^2\times10^{-22}}+\frac{39.478}{137\times4.59\times10^{-8}}\right)\times6.24\times10^{18}\frac{e^{\mu T}-1}{\mu T}=$$

$$=1.7\times10^{-38}\left(1.64\times10^{13}+6.27\times10^{6}\right)\times6.24\times10^{18}\frac{e^{\mu T}-1}{\mu T}\approx1.7\times10^{-38}\times1.64\times10^{13}\times6.24\times10^{18}\frac{e^{\mu T}-1}{\mu T}=17.39\times10^{-7}\frac{e^{\mu T}-1}{\mu T}$$

or $17.39\times10^{-7}\frac{e^{\mu T}-1}{\mu T}$ □ $1{,}56\times10^{5}\Rightarrow\frac{e^{\mu T}-1}{\mu T}=\frac{1{,}56\times10^{5}}{17.39\times10^{-7}}\le8.97\times10^{10}$ which is satisfied for $\mu T=28$.

Then $\frac{e^{\mu T}-1}{\mu T}=5.16\times10^{10}$. To estimate the radius, we use inequalities

$$1.7\times10^{-38}\left(\frac{4.59\times10^{-8}}{\rho_2^{\ 2}}+\frac{39.478}{137\times4.59\times10^{-8}}\right)\times6.24\times10^{18}\times5.16\times10^{10}\le1{,}56\times10^{5}\approx\frac{4.59\times10^{-8}}{\rho_2^{\ 2}}\le2.85\times10^{13}-6.27\times10^{6}\Leftrightarrow$$

$\rho_2\ge0.4\times10^{-10}=4\times10^{-11}m.$

The Obtained Inequality Applied to the Helium-Atom

We recall assumption $\omega_2=\omega_3=\omega,$ and $\omega=4.1\times10^{16}$. Then $T=2\pi/\omega=1.53\times10^{-16}$.
The ionization work for transition from neutral atom to positive atom with a single charge is 24.4 eV, (cf. [9], [10]). Then

$$|E_2(t)|\le\frac{e_{el}^{\ 2}}{137}\left[\frac{4\pi^2}{137cT}+\frac{cT}{\rho_2^{\ 2}}+\frac{cT}{|\rho_{20}-\rho_{30}|^2}+\frac{4\pi}{137|\rho_{20}-\rho_{30}|}\right]\frac{e^{\mu T}-1}{\mu T}\times6.24\times10^{18}\le24.4eV;$$

$$|E_2(t)|\le\frac{2.56\times10^{-38}}{137}\left[\frac{39.478}{137\times4.59\times10^{-8}}+\frac{4.59\times10^{-8}}{5.29^2\times10^{-22}}+\frac{4.59\times10^{-8}}{|\rho_{20}-\rho_{30}|^2}+\frac{12.566}{137|\rho_{20}-\rho_{30}|}\right]7.103\times10^{8}\times6.24\times10^{18}=$$

$$=82.44\times10^{-14}\left[0.0627\times10^{8}+0.16\times10^{14}+\frac{4.59\times10^{-8}}{|\rho_{20}-\rho_{30}|^2}+\frac{0.091}{137|\rho_{20}-\rho_{30}|}\right]\approx$$

$$\approx10^{-14}\left[5.168\times10^{8}+13.19\times10^{14}+\frac{378.39\times10^{-8}}{|\rho_{20}-\rho_{30}|^2}+\frac{7.5}{|\rho_{20}-\rho_{30}|}\right]\approx13.19+\frac{3.7839\times10^{-20}}{|\rho_{20}-\rho_{30}|^2}+\frac{7.5\times10^{-14}}{|\rho_{20}-\rho_{30}|}\le24.4$$

or

$$11.21|\rho_{20}-\rho_{30}|^2 - 7.5\times10^{-14}|\rho_{20}-\rho_{30}| - 3.7839\times10^{-20} \geq 0;$$

$$|\rho_{20}-\rho_{30}| = \frac{7.5\times10^{-14} \pm \sqrt{7.5^2\times10^{-28} + 4\times11.21\times3.7839\times10^{-20}}}{22.42} = \frac{7.5\times10^{-14} \pm 10^{-10}\sqrt{56.25\times10^{-8}+169.67}}{22.42} \approx$$

$$\approx \frac{7.5\times10^{-14} \pm 10^{-9}\times1.3}{22.42} \approx \frac{\pm10^{-9}\times1.3}{22.42} = \pm10^{-11}\times5.79.$$

Therefore, $|\rho_{20}-\rho_{30}| \geq 5.79\times10^{-11}m$.

Similar estimations are valid for $|E_3(t)|$ $\omega_2 \neq \omega_3$ but $\omega_2 \approx \omega_3$.
We want to mention some interesting similar results with different methods [12]-[18].

Conclusion

The interval values of the energy obtained here contain discrete values from quantum mechanics.

We notice that $2.64\times10^{-11} < 3.05\times10^{-11} < 3.74\times10^{-11} < 5.27\times10^{-11}m$. It follows that an atom with more positive charges can have smaller radius, that is, there is a stronger attraction between the nuclei.

Let us calculate the wavelenth $\lambda = \frac{c}{v}$ for the first Lyman series. Indeed, it is known that (cf. [10])

$$\rho_2 = 4\rho_1; \beta_2 = \beta_1/2; v_2 = v_1/8 \Rightarrow \omega_2 = \omega_1/8 .$$

The difference between the energies of the first and second stationary states is

$$E_1(t) - E_2(t) = e_{el}^2\int_0^T\left(\frac{c\beta_1}{\rho_1^2}+\frac{\beta_1^2\omega_1^2}{c}\right)ds - e_{el}^2\int_0^T\left(\frac{c\beta_2}{\rho_2^2}+\frac{\beta_2^2\omega_2^2}{c}\right)ds \leq e_{el}^2\frac{e^{\mu T}-1}{\mu}\left(c\left(\frac{\beta_1}{\rho_1^2}-\frac{\beta_2}{\rho_2^2}\right)+\frac{\beta_1^2\omega_1^2-\beta_2^2\omega_2^2}{c}\right)=$$

$$= e_{el}^2\frac{e^{\mu T}-1}{\mu}\left(c\left(\frac{\beta_1}{\rho_1^2}-\frac{\beta_1}{32\rho_1^2}\right)+\frac{1}{c}\left(\beta_1^2\omega_1^2-\frac{\beta_1^2}{4}\frac{\omega_1^2}{8}\right)\right) = e_{el}^2\frac{e^{\mu T}-1}{\mu}\frac{31}{32}\left(\frac{c}{137\rho_1^2}+\frac{\omega_1^2}{137^2c}\right)=$$

$$= 2.56\times10^{-38}\frac{e^{\mu T}-1}{\mu}0.97\left(\frac{3\times10^8}{137\times5.29^2\times10^{-22}}+\frac{4.1^2\times10^{32}}{137^2\times3\times10^8}\right) \approx \frac{e^{\mu T}-1}{\mu}\times\left(1.9344\times10^{-11}+7.44\times10^{-18}\right) \approx \frac{e^{\mu T}-1}{\mu}\times1.9344\times10^{-11}.$$

The wavelength for the first Lyman series is $\lambda = 1.22\times10^{-7}m$. In view of our result the following question arises: to find the value for $\left(e^{\mu T}-1\right)/\mu$ such that

$$\lambda = \frac{c}{v} = \frac{\hbar c}{E_1-E_2} = \frac{6.62\times10^{-34}\times3\times10^8}{\left(\left(e^{\mu T}-1\right)/\mu\right)\times10^{-11}\times1.9344} = \frac{\mu T}{e^{\mu T}-1}\frac{1.026\times10^{-14}}{T} = 1.22\times10^{-7} \Leftrightarrow \frac{e^{\mu T}-1}{\mu T} = 7.103\times10^8.$$

t is easy to verify that if we choose $\mu T = 23.54$ one obtains $\frac{e^{23.54}-1}{23.54} \simeq 7.103\times10^8$. Therefore, in view of $T = 2\pi/\omega = 1.53\times10^{-16}$ we take $\mu = \frac{23.54}{1.53\times10^{-16}} \approx 1.54\times10^{17}$. Since $\mu T \in (2\pi; 23.54)$ then

$$f(\mu T) = \Delta E_{12} = E_1 - E_2 = \frac{e^{\mu T}-1}{\mu T}\times T\times10^{-11}\times1.9344 : (2\pi; 23.54) \to (E_{\min}; E_{\max}) .$$

Let us recall that we must choose $\mu > \omega \Rightarrow \mu T > \omega T = 2\pi$. The condition $\mu > \omega$ provides the convergence of successive approximations (cf. [2], [4]).

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