ISSN: 3066-8131
Research Article

Science Set Journal of Physics

Quantum Fisher Information of Three-Level Atom Under the Combined Influence of the Non-Linear Kerr Medium and the Stark Effect

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Submitted: 07 April 2025 **Accepted:** 14 April 2025 **Published:** 21 April 2025

https://doi.org/10.63620/MKSSJP.2025.1052

Citation: *Anwar, S. J., Ibrahim, M., & Khan, K. (2025). Quantum Fisher Information of Three-Level Atom Under the Combined Influence of the Non-Linear Kerr Medium and the Stark Effect. Sci Set J of Physics, 4(2), 01-09.*

Abstract

We explore the dynamics of Quantum Fisher Information (QFI) and Von Neumann Entropy (VNE) in three-level stationary and moving Stark-shifted atomic systems influenced by a Nonlinear Kerr Medium (NLKM). QFI, which measures coherence and sensitivity to parameter changes, decreases over time due to decoherence, with slower decay at higher nonlinearity values. VNE, an indicator of quantum entanglement, increases over time, with stronger growth observed at lower nonlinearity value The Stark e¤ect, caused by an external electric field, introduces energy level shifts that result in oscillatory behavior in both QFI and VNE. Stronger electric fields amplify these oscillations, enhancing sensitivity and entanglement dynamics. The system's phase significantly impacts its behavior, with symmetric patterns emerging at zero phase and more complex dynamics occurring at other phase values.

In moving systems, atomic motion interacts with NLKM and Stark e¤ects to produce periodic modulations in QFI and VNE. Lower nonlinearity values lead to pronounced oscillations, reflecting stronger quantum interactions, while higher values stabilize coherence. Peaks in QFI often align with dips in VNE, indicating a trade-o¤ between precision and entanglement. Transient spikes in VNE highlight moments of enhanced quantum entanglement, while high QFI values signify robust coherence, critical for precision measurements. This study demonstrates the tunability of quantum systems through external field strength, nonlinearity, and phase, providing a framework for optimizing quantum sensing and exploring the fundamental dynamics of coherence and entanglement in quantum systems.

Keywords: Quantum Entanglement (QE), Quantum Fisher Information (QFI), Stark E¤ect (SE), von Neumann entropy (VNE), Non-Linear Kerr Medium (NLKM)

Introduction

The interaction between a two-level atom and a single quantized mode of a radiation field, within the framework of the rotating wave approximation (RWA), is commonly referred to as the Jaynes-Cummings model (JCM) [1]. This model describes the cyclic process of photon absorption and emission by the atom, which enables energy and momentum transfer between the atom and the radiation field. Studying the JCM provides valuable insights into the dynamics of atom-field interactions. The JCM has been the subject of extensive theoretical [2] and experimental investigations [3]. Over the past few decades, quantum optics research has increasingly focused on three-level systems interacting with one- or two-mode cavity fields in various configurations. Yoo and Eberly conducted in-depth studies of three-level atoms in di¤erent configurations interacting with quantized fields under RWA in an ideal cavity [4].

A generalized framework for the JCM, involving three-level atoms interacting with one- or two-mode non-resonant cavity fields, has also been established [5]. Additional studies have examined three-level systems under multi-photon interactions, systems with field-dependent coupling constants, NLKM and

interactions involving non- correlated two-mode fields [6-10]. Recent advancements include the application of the modified homotopy analysis method to obtain the wave function of a three-level atom and the derivation of a general solution for such systems in the presence of gravity [11, 12]. Other areas of interest include analyzing atomic motion in the JCM, investigating the dynamics of entanglement in three-level atoms, and studying interactions between a three-level atom in a momentum eigenstate and a one-mode cavity field within a NLKM [13-15]. Furthermore, alternative Lie algebra SU (3) approaches have been employed to explore the internal and external dynamics of a three-level atom interacting with a one-mode cavity field under the influence of an atomic gravity field [16].

When particles are separated by a certain distance, they can exhibit a remarkable con- nection that persists regardless of the physical separation between them. This phenomenon, known as QE, allows the state of one particle to instantaneously a¤ect the state of another, even when they are far apart [17-20]. In real-world quantum systems, open quantum structures are critical to consider. These systems interact with their environment and are influenced by external conditions, making them highly relevant in understanding entangle- ment dynamics. However, such interactions often lead to detrimental e¤ects, including a process known as decoherence. Decoherence refers to the degradation of coherence or en- tanglement within the system, which ultimately results in the loss of quantum information [21-23].

To fully comprehend the impact of environmental factors on quantum information, it is vi- tal to evaluate the parameters of a quantum system both before and after its interaction with a noisy channel. This evaluation process, known as quantum metrology, focuses on utilizing quantum principles to enhance measurement precision [24, 25]. Quantum metrology encom- passes various techniques, such as the Quantum Cramer-Rao Bound, QFI, Heisenberg-Limit Metrology, and Time-Evolution-Based Metrology. Among these, QFI serves as a founda- tional metric for determining the precision of parameter estimation in quantum systems, highlighting its versatility in comparing di¤erent metrological methods [26].

QFI plays a pivotal role in quantum metrology by defining the theoretical limits of parameter estima- tion precision for quantum states. While much research has been devoted to exploring QFI independently of noisy environments [27, 28], the substantial influence of dissipative processes—prevalent in real-world quantum systems—warrants significant attention. Dissi- pative channels, which induce decoherence and result in information loss, are particularly critical in this regard [29, 30]. Kraus operators o¤er a powerful mathematical framework for modeling open quantum systems, providing insights into how dissipative interactions modify quantum states and a¤ect QFI [31]. This framework is essential for understanding the be- havior of quantum systems in practical scenarios, where noisy and dissipative environments are unavoidable.

The Kerr e¤ect refers to the phenomenon where the refractive index of a material changes in response to an applied electric field, leading to a nonlinear optical e¤ect When combined with quantum entanglement, the Kerr e¤ect can be used to create and manipulate entangled states [32]. For instance, in an optomechanical system, the Kerr e¤ect can enhance the entanglement between light and mechanical oscillators [33]. The Stark e¤ect occurs when an external electric field causes a shift and splitting of atomic or molecular energy levels [34]. This e¤ect can be used in quantum information processing to control and generate entangled states [35]. For example, the dynamic Stark e¤ect can be employed to generate entanglement between photons and atoms in a laser-driven quantum optical system [36].

The goal of our current investigation is to determine the QE dynamics of a three-level atomic system under the influence of the SE and NLKM, considering both the presence and absence of a moving atom. Our motivation lies in examining how these e¤ects impact QE. We investigate how both the NLKM) and SE simultaneously a¤ect the dynamics of QFI) and VNE in three-level atomic systems, considering both stationary and moving atoms.

It is evident the SS predominate during the quantum system's temporal growth. The SS significantly a¤ects QFI dynamics. Furthermore, the SS a¤ects the VNE more strongly when there is no motion of atom. VNE is a crucial concept in quantum statistical mechanics, where it helps describe the thermodynamic properties of quantum systems in equilibrium. It relates to concepts like temperature and free energy in the quantum context.

The structure of this document is organized as follows: Section 2 outlines the system model, focusing on a three-level atomic system influenced by a coherent field, incorporat- ing the e¤ects of SE and NLKM. This section also includes the model Hamiltonian and interaction dynamics. Section 3 presents a detailed discussion along with numerical results. Finally, Section 4 concludes with a concise summary of the findings.

Hamiltonian Model

We study the system of moving 3-level atom interacting with coherent field in the presence of combined e¤ect of the SE and NLKM we study the cascade configuration of the system. Assuming the RWA, the system H^T , is given by [37].

$$\hat{H}_T = \hat{H}_{Atom-Field} + \hat{H}_I. \qquad (1)$$

where $H_{Atom\text{-}Field}$ shows the atom which is not interacting and field Hamiltonian, and H_I represents the coupling portion. Our writing for $H_{Atom\text{-}Field}$ will be

$$\hat{H}_{Atom-Field} = \sum_j \omega_j \hat{\sigma}_{j,j} + \nu \hat{a}^\dagger \hat{a}, \qquad (2)$$

where the jth levels σj,j = |j j| represents atomic population operator. atomic popula- tion operators, were essentially referring to the operators that describe transitions between di¤erent energy levels of the atom. These transitions play a crucial role in entanglement dynamics.

It depicts a two-level atom when N = 1, and the 3-, 4-, and 5-level atoms when N = 3, 4, and 5.

When it comes to SE and NLKM, the H^I is provided by

$$\hat{H}_I = \sum_{s=1}^{3} \Omega(t) \left[\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} + \left(\hat{a} e^{-i\Delta_s t} \hat{\sigma}_{s,s+1} \right)^{\dagger} \right] + \beta \hat{a}^{\dagger} \hat{a} + \chi \hat{a}^{\dagger 2} \hat{a}^2, \tag{3}$$

χ is the parameter of NLKM, β is the parameter of SE and and the detuning parameter is described as

$$\Delta_s = \nu - (\omega_s - \omega_{s+1}). \tag{4}$$

The coupling constant between the atom and the field is denoted as g, while (t) represents the shape function of the cavity-field mode [37]. The atom's motion is assumed to occur along the z-axis for specific considerations of interest.

(t) = g sin (pvvt/L) in the presence of atomic motion , p /= 0
(t) = g in the absence of atomic motion p = 0 (5)

The velocity of the atom is represented by v, p corresponds to half the number of wavelengths of the cavity mode, and L denotes the cavity's length along the z-direction. By setting the atom's velocity as v = gL/v, we arrive at the following result.

$$\Omega_1(t) = \int_0^t \Omega(\tau) d\tau = \frac{1}{p}(1 - \cos(p\pi vt/L) \text{ for } p \neq 0 \tag{6}$$

$$= gt \text{ for } p = 0. \tag{7}$$

To achieve maximum precision in determining the phase shift parameter, the optimal input state is considered as follows

$$|\Psi(0)\rangle_{\text{Opt}} = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle) \otimes |\alpha\rangle. \tag{8}$$

Here, |1and |0 represent the states of the atom, while α denotes the coherent state of the input field, expressed as follows

$$|\alpha\rangle = \sum_{n=0}^{\infty} \alpha^n \sqrt{e^{-|\alpha|^2}/n!} |n\rangle \tag{9}$$

We analyze a phase gate that operates on a single atom and applies a phase shift

$$\hat{U}_{\phi} = |1\rangle\langle 1| + e^{i\phi}|0\rangle\langle|0|, \tag{10}$$

The state |W(0) is derived by applying the single-atom phase gate to the initial optical state
|W(0) Opt

$$\hat{U}_{\phi}|\Psi(0)\rangle_{\text{Opt}} = |\Psi(0)\rangle \tag{11}$$

$$= \frac{1}{\sqrt{2}}(|1\rangle + e^{i\phi}|0\rangle) \otimes |\alpha\rangle \tag{12}$$

Following the phase gate operation, the system interacts with a field. In the time- independent scenario, the wave function is described using the transformation matrix U^ (t)
as expressed

$$|\Psi(t)\rangle = \hat{U}(t)|\Psi(0)\rangle \tag{13}$$

The density matrix (DM) can be explicitly expressed as

$$\hat{\rho}(t) = \sum_{m,n}^{N} |\Psi_n(t)\rangle\langle\Psi_n(t)|\hat{\rho}(0)|\Psi_m(t)\rangle\langle\Psi_m(t)| \tag{14}$$

In this manner, the QFI for a bipartite density operator ρAB. can be defined with respect to , as described by [38].

$$I_{QF}(t) = I(\phi, t) = \text{Tr}[\rho_{AB}(\phi, t)\{L^2(\phi, t)\}], \tag{15}$$

here, L(, t) represents the quantum score, also known as the symmetric logarithmic deriv- ative (SLD), and can be determined as described by [39].

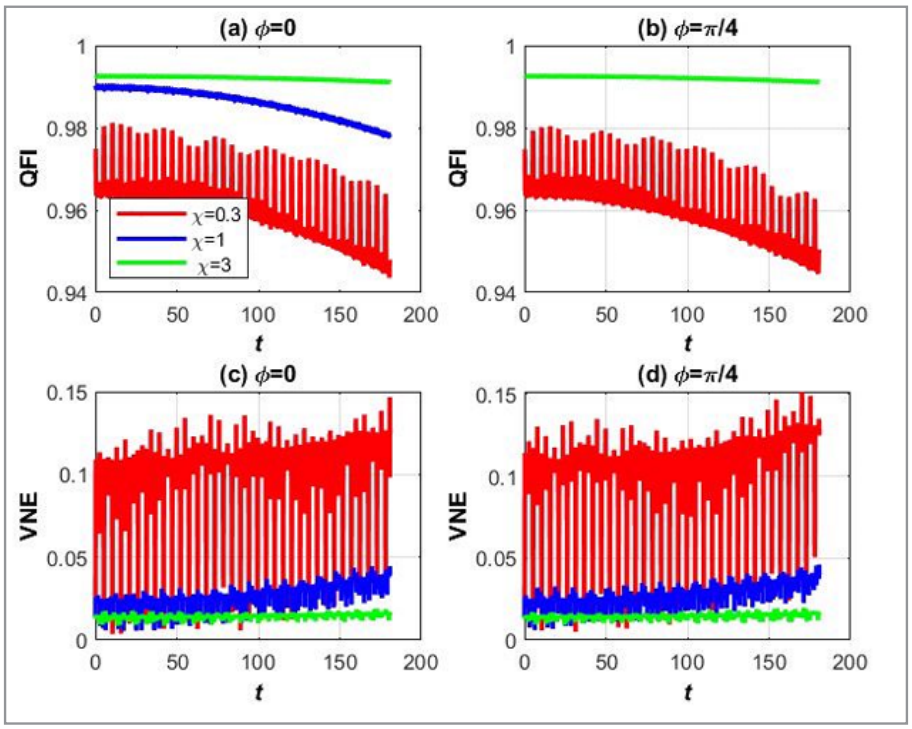
$$\frac{\partial\rho(\phi, t)}{\partial\phi} = \frac{1}{2}[L(\phi, t)\rho_{AB}(\phi, t) + \rho_{AB}(\phi, t)L(\phi, t)]. \tag{16}$$

Likewise, the VNE is described as

$$S_A = -\text{Tr}(\rho_A \ln \rho_A) = -\sum_i r_i \ln r_i, \tag{17}$$

where ri represent eigenvalues of the atomic DM ρ_A =Trв(ρ_{AB}). The next section explores how various environmental parameters , p, χ, and β a¤ect the evolution of the QFI and VNE.

We have done the numerical calculations as the system is bigger to solve analytically, we used computational languages and software for the numerical calculations. We have described clearly the Hamiltonian of the system and the interaction Hamiltonian. We also mention the state and also the formulas for QE quantifiers like QFI and VNE. To quantify QE, we construct the density matrix, and this density matrix is solved numerically. With the help of this density matrix Eigen vectors and Eigen values of the density matrix are calculated. These eigenvectors and eigenvalues are used to calculate QFI and VNE but this whole calculation is numerical.



The QFI (upper portion) and VNE (lower portion) vs time, α = 6, β = 0.3 and χ = 0.3, 1,3 and = 0 (left side) and v/4 (right side), p = 0.

Discussions and Numerical Outcomes

We consider both of the static and moving cases of the three- level system, and the presence of the coherent field. The systems level is SE with strength β in the presence of NLKM. We solve the system dynamics numerically and have chosen a 0.1-time step size.

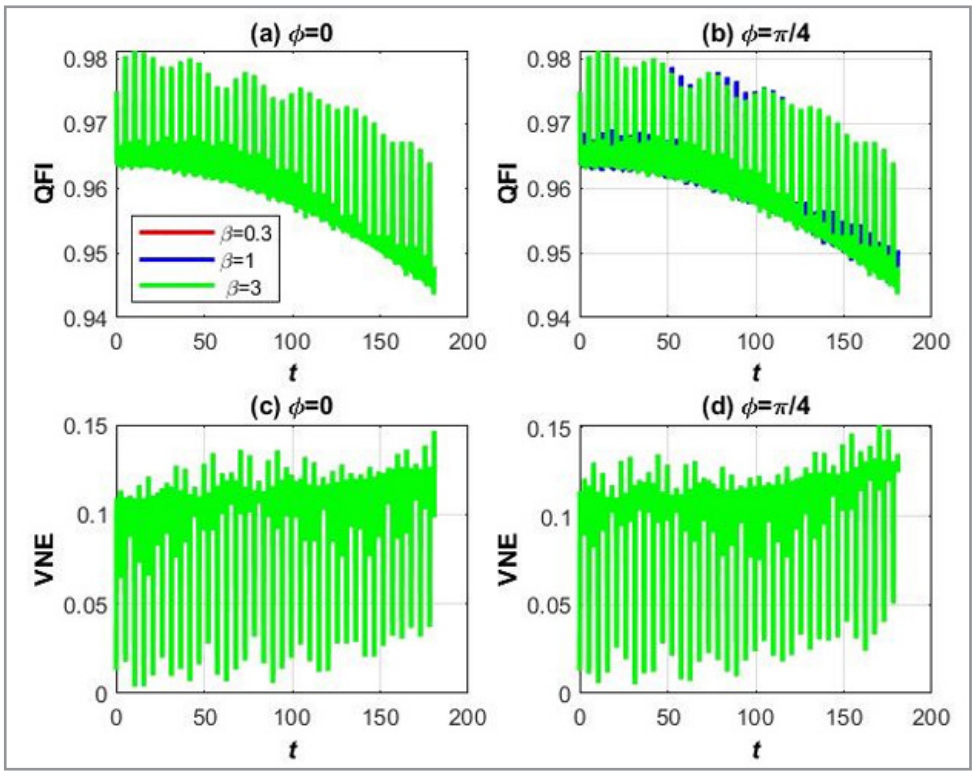


Figure 1: The QFI (upper portion) and VNE (lower portion) vs time, = 6, = 0.3 and β = 0.3 1,3 and = 0 (left side) and π/4 (right side), p = 0.

VNE and QFI of three- level stationary Stark Shifted atomic systems under the Influence of NLKM

Explanation of the Graphs

Fig-1 QFI measures the information that an observable random variable carries about an unknown parameter, used to quantify the sensitivity of a quantum state to changes in a parameter. Subplots (a) and (b) show QFI versus time for two phase values: = 0 and = v/4. Each curve represents di¤erent NLKM parameter values: χ = 0.3 (red), 1 (blue), and 3 (green). QFI decreases over time for all 3 values, with a more pronounced decrease for lower χ values. VNE quantifies the amount of quantum entanglement (QE) in a state. Subplots (c) and (d) show VNE versus time for the same phase alues. Each curve represents di¤erent χ values: 0.3 (red), 1 (blue), and 3 (green). VNE increases over time for all χ values, with a more pronounced increase for lower χ values. The Stark e¤ect refers to the shifting and splitting of spectral lines due to an external electric field (β = 0.3). NLKM parameter

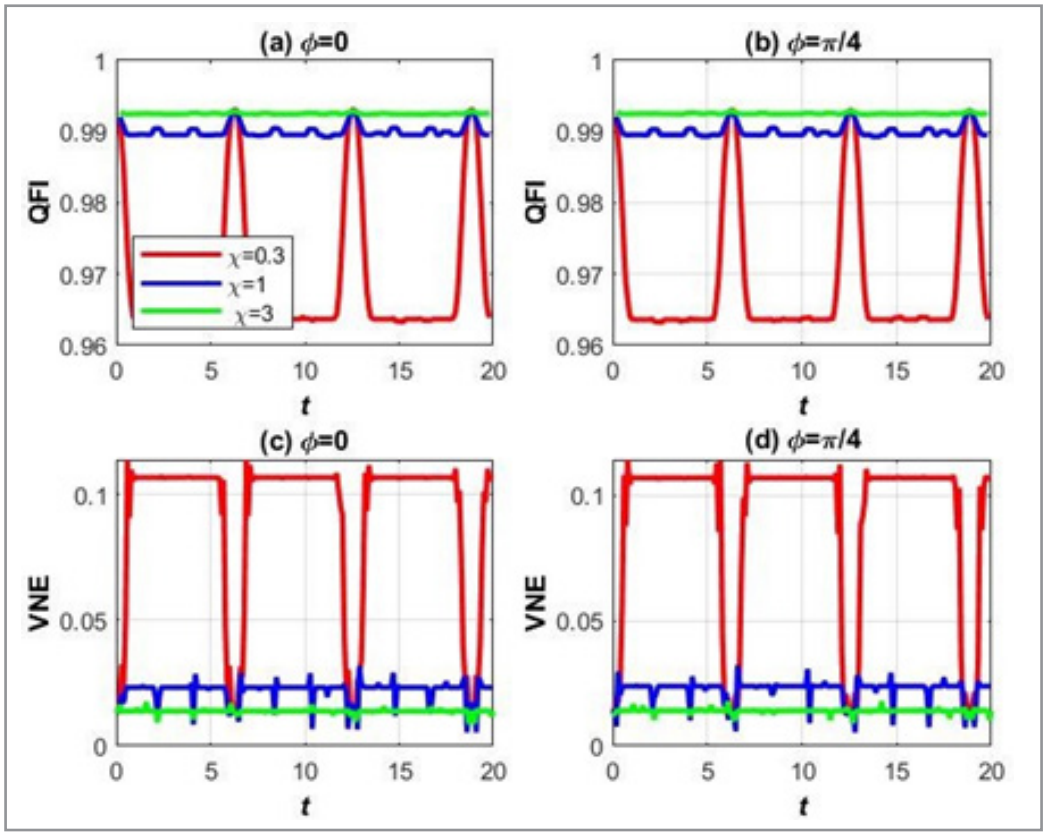


Figure 2: The QFI (upper portion) and VNE (lower portion) vs time, = 6, β = 0.3 and = 0.3 1,3 and = 0 (left side) and π/4 (right side), p = 1.

χ represents the strength of non-linear interaction. Lower χ values lead to a higher increase in VNE, indicating stronger QE, while higher χ values result in a slower increase, indicating weaker QE. QFI is crucial in quantum metrology for estimating a parameter, with higher QFI indicating more extractable information. Higher values stabilize the quantum state against decoherence, slowing QFI decay over time. The Stark e¤ect (β = 0.3) influences energy levels and phases, a¤ecting sensitivity to parameter changes. VNE measures the degree of entanglement. Lower χ values enhance QE, leading to a steeper increase in VNE, while higher χ values result in slower QE dynamics. The applied electric field (β = 0.3) shifts energy levels and alters phase, influencing entanglement generation and evolution by changing energy state interactions. QFI decreases over time due to decoherence, with a faster decrease for lower χ values and a slower decrease for higher χ values. VNE increases over time, indicating growing QE, with a faster increase for lower χ values and a slower increase for higher χ values. At = 0, the initial phase doesn't introduce additional complex phases, and the decrease in QFI is driven by the systems intrinsic dynamics. At

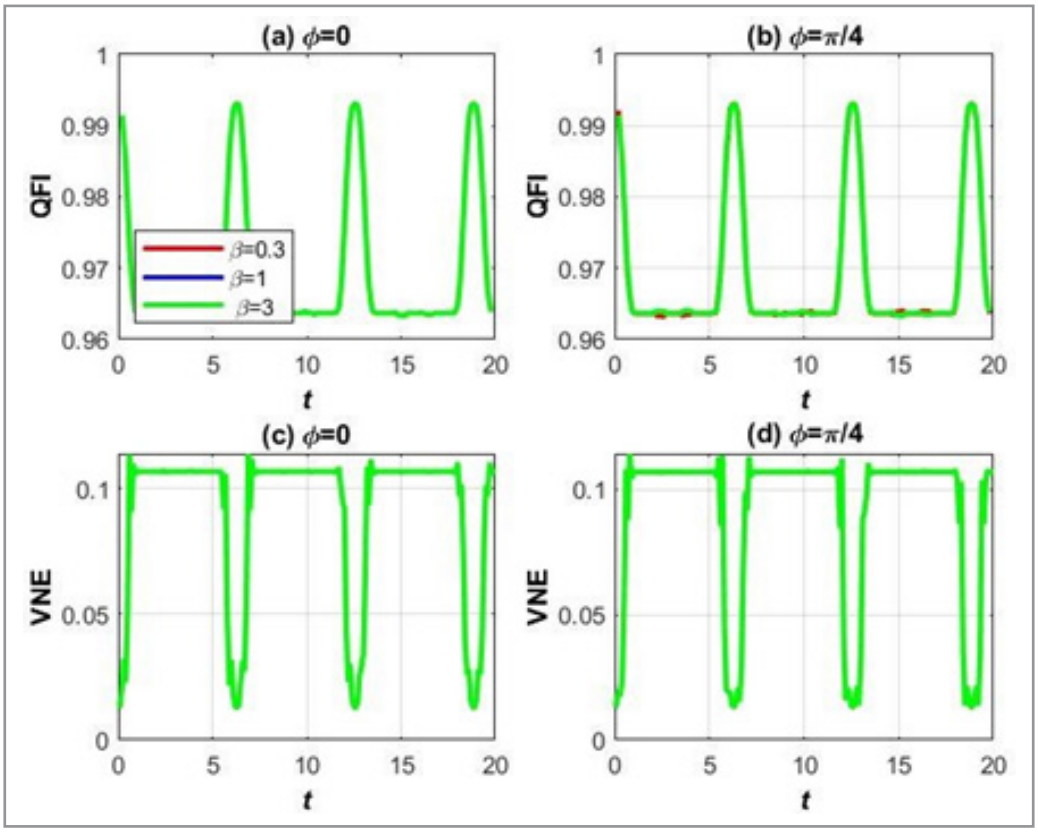


Figure 3: The QFI (upper portion) and VNE (lower portion) vs time, = 6, = 0.3 and β = 0.3 1,3 and = 0 (left side) and π/4 (right side), p = 1.

v/4, additional phase shifts a¤ect interference patterns and sensitivity. The Stark e¤ect and NLKM interactions shape QE and information metrics, with phase introducing complexity into the system's evolution.

FIG-2 QFI measures the sensitivity of a quantum state to parameter changes and is crucial for quantum metrology. Higher QFI indicates better precision. VNE measures quantum entanglement (QE); higher VNE indicates more QE.

The Kerr e¤ect is a nonlinear optical e¤ect where a material's refractive index changes with light intensity. The parameter χ represents this nonlinearity. The Stark e¤ect refers to spectral line shifts due to an external electric field, with parameter β indicating its strength. Subplot (a) = 0, β = 0.3, 1, and 3, QFI starts around 0.98 and decreases over time, with more significant fluctuations for higher β. Subplot (b) = v/4: Similar to (a) but with a slightly more pronounced QFI decrease. Subplot (c) = 0: VNE fluctuates around 0.1, with larger fluctuations for higher β. Subplot (d) = v/4: Similar to (c) but with slightly more pronounced fluctuations. The Stark e¤ect leads to significant fluctuations in both QFI and VNE, indicating its impact on sensitivity and QE dynamics. This analysis helps understand how external fields and nonlinear e¤ects influence precision and QE in quantum systems. QE, where particles remain connected regardless of distance, is vital for quantum computing and cryptography. QFI indicates how much information a quantum system carries about a parameter. Entangled states generally have higher QFI, improving parameter estimation precision. In quantum metrology, entangled states achieve precision beyond classical limits. QE dynamics, influenced by external fields and nonlinear interactions, a¤ect QFI.

VNE measures QE, with higher values indicating more entanglement. Nonlinearities and external fields cause QE to fluctuate, often correlating with QFI changes. Both QFI and VNE are linked to QE. High QFI often corresponds to high VNE. The Kerr nonlinearity and Stark e¤ect modulate QE, causing fluctuations in both measures. Understanding these relationships optimizes quantum systems for quantum metrology and information processing. Graphs show QFI and VNE varying over time and with di¤erent β values. Higher β results in more significant QFI and VNE changes, indicating stronger external fields enhance QE and sensitivity. Utilizing highly entangled states with high QFI allows precision measurements beyond classical limits. Analyzing QE and sensitivity changes with external parameters tailor's quantum systems for specific tasks.

VNE and QFI of Three- Level Moving Stark Shifted Atomic Systems under the Influence of NLKM

Explanation of Graphs

FIG-3 This graph depicts the behavior of QFI and VNE as a function of time under di¤erent conditions of the Kerr nonlinearity parameter χ = 0.3, 1 and 3 and the phase = 0, = v/4. The

key features are influenced by atomic motion, the Stark e¤ect parameter β = 0.3, and QE in NLKM. (a) QFI for = 0 and (b) QFI for = v/4: QFI measures the sensitivity of a quantum state to changes in a parameter and is directly related to the precision of quantum measurements. Higher QFI values imply better precision. For χ = 0.3 (red line), QFI shows periodic oscillations with sharp dips, indicating significant influence of the NLKM and atomic motion. For χ = 1 (blue line) and χ = 3 (green line), QFI stabilizes at high values, reflecting lesser sensitivity to the atomic motion or nonlinear e¤ects. (c)

VNE for = 0 and (d) VNE for = v/4: VNE quantifies QE, with higher values indicating stronger QE. The red line (χ = 0.3) exhibits pronounced oscillations in VNE, showing strong variations in QE over time, corresponding to periodic coupling between subsystems mediated by the NLKM and atomic motion. For χ = 1 and χ = 3, VNE remains relatively low, indicating weaker QE. The oscillatory behavior in both QFI and VNE, especially for χ = 0.3, is indicative of dynamic interactions between atomic motion and the NLKM. This motion induces periodic modulations in QE and precision QFI, particularly at low χ. QE plays a pivotal role in enhancing QFI. Peaks in QFI often align with dips in VNE, suggesting a trade-o¤ between the precision of parameter estimation.

FIG-4 Atoms in this system are coupled to a nonlinear medium (Kerr medium) and experience periodic dynamics. The motion could correspond to oscillations between energy levels due to Rabi cycling or interactions with an external field. These oscillations are re- flected in the QFI and VNE's periodic nature. The atomic motion drives transitions between di¤erent quantum states, a¤ecting coherence (QFI) and entanglement (VNE). The NLKM χ = 0.3 introduces a non-linear term into the Hamiltonian, meaning that the interaction strength between the atoms depends on their quantum state. This non-linearity leads to state-dependent phase shifts that influence the atomic dynamics. Imagine that the atomic system is in a cavity where the non-linearity of the medium bends the "optical paths" of the atoms in a way that depends on their quantum states.

This bending causes oscillatory be- heavier in the quantum coherence and QE. The oscillations in QFI and VNE emerge from the interplay between the Kerr-induced non-linearity and the periodic atomic motion. QFI as a Measure of Coherence: QFI reflects how much information about a arameter (like) can be extracted from the quantum state. A high QFI indicates strong coherence and sensitivity to external parameters. High QFI values (m0.96): The system is largely coherent, meaning the quantum state retains well-defined phase relationships. Periodic Dips: As the atomic motion evolves, interactions in the Kerr medium and external fields periodically modulate the state coherence, causing temporary reductions in QFI.

Phase Dependence (— 0 vs. = v/4): Changing adjusts the interference pattern of the atomic wavefunctions, slightly shifting the oscillations. VNE quantifies the QE or mixedness of the system. If the entropy is low (close to zero), the system is in a pure state; higher values indicate more QE or decoherence. Low QE (—0.1): The system is predominantly in a pure quantum state, suggesting minimal interaction between subsystem. The spikes in VNE correspond to moments when the atomicmotion and non-linear interactions temporarily increase QE between di¤erent parts of the system (e.g., between atomic internal states or field modes in the Kerr medium). The Stark e¤ect β = 0.3, 1 and 3 shifts atomic energy levels due to an external electric field.

The para- meter β quantifies the strength of this shift. Physically, β modifies the time evolution of the system by introducing energy level di¤erences.: Small β = 0.3: The energy shifts are small, so the atomic motion and coherence oscillations are less disturbed. This results in smoother dynamics, as seen in the red curves. Large β = 3: Higher Stark shifts introduce stronger energy level splitting, leading to faster and sharper oscillations in both QFI and VNE. The green curves reflect these more pronounced dynamics. The atomic system evolves under the competing influences of the Kerr medium's non-linearity and the Stark e¤ect. Physically, this is akin to a pendulum swinging in a medium that changes its properties depending on the pendulum's state. The high QFI indicates the pendulum's motion remains regular and predictable, making the system highly sensitive to phase or energy shifts.

Small spikes in VNE represent transient moments when the pendulums motion briefly entangles with other degrees of freedom, such as the medium or another pendulum. Changing adjusts the initial conditions of the systems dynamics, similar to starting a pendulum swing from di¤erent angles. In the graphs at = 0: the oscillations appear slightly simpler and more sym- metric. t = v/4, a phase o¤set introduces subtle asymmetries or shifts in the oscillation patterns of QFI and VNE. Atomic motion: Drives periodic coherence and QE dynamics. NLKM medium (χ = 0.3) introduces non-linearity, leading to periodic modulation of quan- tum properties.

The SE β alters energy levels, influencing the amplitude and frequency of the oscillations. QFI shows high coherence, suitable for quantum metrology. VNE: Shows minimal QEt, indicating a predominantly pure state with transient increases in QE. This interplay demonstrates a quantum system where coherence dominates, with QE emerging briefly during atomic motion cycles. The physical system is highly controllable via β, χ, and , making it ideal for precision quantum sensing or fundamental studies of quantum dynamics.

Conclusion

The study investigates the behavior of QFI and VNE in three-level stationary and mov- ing Stark-shifted atomic systems coupled with a nonlinear Kerr medium. QFI, representing quantum coherence and sensitivity, decreases over time due to decoherence but stabilizes for higher Kerr nonlinearity values. VNE, a measure of quantum entanglement, increases over time, with stronger growth observed for lower nonlinearity values, reflecting enhanced entanglement. The Stark e¤ect introduces energy level shifts that modulate quantum state dynamics, leading to oscillatory behavior in both QFI and VNE. The phase parameter influences these oscillations, with symmetric patterns at one phase value and more complex dynamics at another.

In the moving systems, the atomic motion interacts with the nonlinear medium and the Stark e¤ect, creating periodic oscillations in QFI and VNE. For lower Kerr nonlinearity values, these oscillations are more pronounced, indicating a stronger impact

of nonlinear interactions and motion on coherence and entanglement. Higher nonlinearity val- ues stabilize QFI, while VNE exhibits reduced fluctuations, indicating weaker entanglement. The Stark e¤ect significantly a¤ects the oscillation amplitude and frequency, with stronger energy shifts leading to sharper fluctuations. Changes in the phase parameter further modu- late interference patterns and the evolution of QFI and VNE.

Peaks in QFI often align with dips in VNE, suggesting a trade-o¤ between coherence and entanglement. The interplay of Kerr nonlinearity, Stark e¤ect, and phase provides a controllable framework for managing quantum properties such as coherence and entanglement. Lower nonlinearity values enhance entanglement, while higher values preserve coherence, making these systems suitable for pre- cision quantum metrology. The ability to control the system parameters allows for tailored quantum systems designed for applications in sensing, quantum computing, and studies of fundamental quantum dynamics.

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