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Analytical and Numerical Solutions of Partial Differential Equations with Emphasis on Finite Difference Methods

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Abstract

This study explores the analytical and numerical solutions of partial differential equations (PDEs), focusing on hyperbolic the first part presents their analytical solutions using initial and boundary conditions and delves into the finite difference method (FDM), discussing forward, backward, and central difference schemes. These methods are applied to numerically solve one- and two-dimensional heat. The Crank-Nicolson method, recognized for its unconditional stability, is employed to improve the accuracy of wave equation solutions, overcoming limitations of explicit and implicit schemes. We then analyze the performance, strengths, and weaknesses of FDM through numerical simulations of one-dimensional heat. Due to computational constraints, Crank-Nicolson for 1Dsimulation, was not executed. Results indicate that the implicit backward difference method demonstrates superior stability by allowing unrestricted step sizes compared to the explicit forward difference method. These findings contribute to a deeper understanding of numerical PDE solutions and stability considerations in computational mathematics.

Keywords: PDEs, hyperbolic (heat) Equation, Crank-Nicolson Method

Introduction

Many problems arising in physical phenomena such as those found in physics, applied sciences, and engineering can be effectively modeled using partial differential equations (PDEs) [1]. A PDE involves a function of two or more independent variables, often including time and spatial coordinates. Compared to ordinary differential equations, PDEs pose greater complexity due to their dependence on multiple variables, often leading to challenging numerical tasks. Solving such equations typically requires advanced scientific computation, often facilitated by modern computing systems [2]. PDEs serve as the mathematical foundation for numerous physical and engineering processes, including electrostatics, heat conduction, fluid dynamics, electrodynamics, and gravitational potential [3]. They also underpin models in chemical and biological systems, and their applications have recently extended to fields such as economics, financial forecasting, and image processing [4]. Boundary conditions play a critical role in solving PDEs, particularly in equations like the heat and wave equations. For the heat equation, initial values are defined within a bounded domain, with the Dirichlet condition typically applied for positive time values. In contrast, the classical boundary condition for the wave equation is the Cauchy problem, which specifies both initial position and velocity. Determining the appropriateness of boundary conditions for a given PDE can be complex and often requires fundamental mathematical insight. The development of finite difference methods in the 1950s, alongside the rise of computational systems, provided powerful tools for numerically solving PDEs. Over the last several decades, significant theoretical advancements have been made regarding the accuracy, consistency, stability, and convergence of these methods [5]. Finite difference methods work by discretizing the domain and approximating PDEs with systems of algebraic equations. Notably, these methods are among the oldest and simplest techniques for solving PDEs, with origins tracing back to L. Euler (ca. 1768) for one-dimensional problems and later extended to two dimensions by C. Runge (ca. 1908).

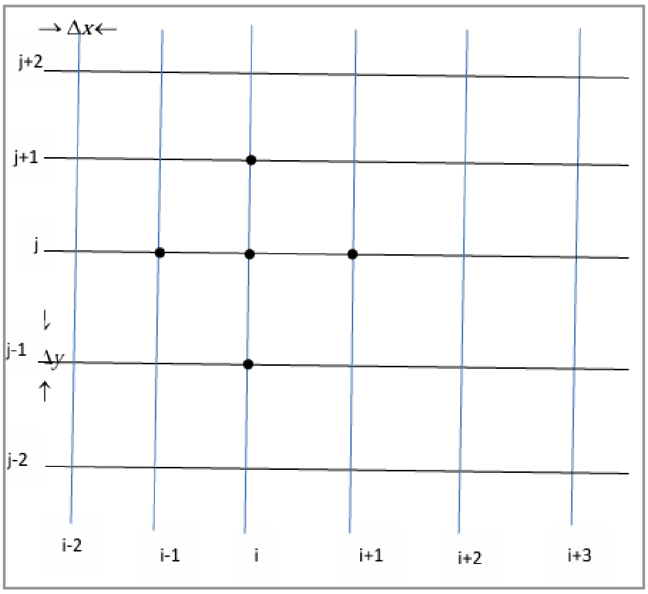

Figure 1.1

Analytical Solutions to Hyperbolic Equation

One-dimensional solution of the wave equation

The wave equation is used to model the displacement of an elastic string or the longitudinal vibration of beam, shown below:

$$\frac{\partial^2}{\partial t^2}u(x,t)=c^2\left(\frac{\partial^2}{\partial x^2}u(x,t)\right) \tag{1.26}$$

Where, as in the case of elastic string, and is the mass per unit length of the string. For the case of longitudinal beam, and T be the constant tension in the string, where g is the acceleration due to gravity, E is the modulus of elasticity and is the density of the beam.

Suppose we solve the wave equation for a vibrating string of length L using separation variables with the boundary conditions Also, initial shape of the string is and the initial speed of the is at we postulate a solution of the form as an ODE, and replace this result in original PDE:

$$X(x)\left(\frac{\partial^2}{\partial t^2}T(t)\right)=c^2\left(\frac{\partial^2}{\partial x^2}X(x)\right)T(t) \tag{1.27}$$

Dividing both sides of the equation by , we have:

$$\frac{\frac{\partial^2}{\partial t^2}T(t)}{T(t)}=\frac{c^2\left(\frac{\partial^2}{\partial x^2}X(x)\right)}{X(x)} \tag{1.28}$$

This can only be possible if both the equations are equal to a constant, say, Having this, we write two ODEs, one for each side of the wave equation Eqn1, i.e.,

$$\left(\frac{\partial^2}{\partial t^2}T(t)\right)+\alpha^2T(t)=0 \tag{1.29a}$$

$$\left(\frac{\partial^2}{\partial x^2}X(x)\right)+\alpha^2X(x)=0 \tag{1.29b}$$

The solutions to these equations are:

$$T(t)=\_C1\cos(\alpha,t)+\_C2\sin(\alpha,t) \tag{1.29c}$$

$$X(x)=\_c1\sin\left(\frac{\alpha,x}{c}\right)+\_C2\cos\left(\frac{\alpha,x}{c}\right) \tag{1.29d}$$

The boundary conditions translate into With these boundary conditions; we can have the following algebraic equation:

$$\_C2=0\ ,\quad 0=\_c1\sin\left(\frac{\alpha,L}{c}\right)$$

We have Eigen equation given by the solution to this equation or using only the positive values; we can write . Next, using the value of and we obtain an expression for as follows:

$$\alpha=\frac{n\pi c}{L} \tag{1.29e}$$

$$Sol:=X(x)=\sin\left(\frac{n\pi x}{L}\right) \tag{1.29f}$$

The function is written as:

$$T(t)=\_c1\cos\left(\frac{n\pi ct}{L}\right)+\_C2\sin\left(\frac{n\pi ct}{L}\right) \tag{1.30}$$

$$u(x,t)=\sin\left(\frac{n\pi x}{L}\right)\left(\_C1\cos\left(\frac{n\pi ct}{L}\right)+\_C2\sin\left(\frac{n\pi ct}{L}\right)\right) \tag{1.31}$$

Application of the initial conditions provides the following equations:

$$f(x)=\_C1\sin\left(\frac{n\pi x}{L}\right) \tag{1.32}$$

$$g(x)=\frac{\sin\left(\frac{n\pi x}{L}\right)\_C2\,n\pi c}{L} \tag{1.33}$$

The equations defining the coefficients and are given by:

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \text{ and } b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx, \text{ for } n=1,2,3,..$$

The final solution is, therefore,

$$u(x,t) = \sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{L}\right)\left(a_n\cos\left(\frac{n\pi ct}{L}\right)+b_n\sin\left(\frac{n\pi ct}{L}\right)\right) \quad (1.33)$$

Finie Difference Method for Wave Equation

Consider the FDM for one dimensional wave equation in the equation (1.26) with the initial and boundary conditions and respectively. The FDM operates on the grid as in the fig (2.12) with the same case of parabolic equation. The grid points are , where and , for step sizes and . We will also approximate the solution.

To discretize the wave equation, the second-order partial differential derivatives will the central difference formula as follows:

$$\frac{v(i,j+1)-2v(i,j)+v(i,j-1)}{k^2} - c^2\left(\frac{v(i-1,j)-2v(i,j)+v(i+1,j)}{h^2}\right)=0 \quad (4.3a)$$

By rearranging we have:

$$v(i,j+1)-2v(i,j)+v(i,j-1)=\frac{k^2c^2}{h^2}\left(v(i-1,j)-2v(i,j)+v(i+1,j)\right) \quad (4.3b)$$

Where , and equation (4.3b) becomes:

$$v(i,j+1)-2v(i,j)+v(i,j-1)=\gamma^2\left(v(i-1,j)-2v(i,j)+v(i+1,j)\right) \quad (4.4a)$$

Rearranging the equation (4.4a) will yield:

$$v(i,j+1) = 2v(i,j)-v(i,j-1)+\gamma^2 v(i-1,j)-2\gamma^2 v(i,j)+\gamma^2 v(i+1,j) \quad (4.4b)$$

This implies that equation (4.4b) gives:

$$v(i,j+1) = \gamma^2 v(i-1,j)+\left(2-2\gamma^2\right)v(i,j)+\gamma^2 v(i+1,j)-v(i,j-1) \quad (4.5a)$$

Hence the equation (4.5) becomes:

$$v(i,j+1) = 2\left(1-\gamma^2\right)v(i,j)+\gamma^2\left(v(i+1,j)+v\gamma^2(i-1,j)\right)-v(i,j-1) \quad (4.5b)$$

This equation (4.5) holds for , but cannot be used for the first time step since are needed. But it can be solved by three points cantered difference formula to approximate the first derivative of the solution :

$$u_t(x,t) \approx \frac{v(i,j+1)-v(i,j-1)}{2k} \quad (4.6)$$

Substituting the initial data time step produce

$$f(x_i) = u(x_i,t_0) \approx \frac{v(i,1)-v(i,-1)}{2k} \quad (4.7)$$

In other form

$$v(i,-1) \approx v(i,1)-2kf(x_i) \quad (4.8)$$

Substituting equation (4.8) in to the finite difference formula of (4.5b) for gives

$$v(i,1) = \left(2-2\gamma^2\right)v(i,0)+\gamma^2 v(i-1,0)+\gamma^2 v(i+1,0)-v(i,1)+2kf(x_i), \quad (4.9)$$

That can be solved for to yield

$$v(i,1) = \left(1-\gamma^2\right)v(i,0)+kf(x_i)+\frac{\gamma^2}{2}\left[v(i-1,0)+v(i+1,0)\right] \quad (5.0)$$

So, the formula (5.0) is employed for the first time step. This is where the initial velocity information comes into the calculation. The equation (4.5b) caters for all time steps. Since the second-order formula have been used space and time derivatives, the error of this FDM is going to be and .

The Finite Difference Method of (5.0) In Matrix Form is Defined As

$$A = \begin{bmatrix} 2+2\gamma^2 & \gamma^2 & 0 & 0 & 0 \\ \gamma^2 & 2+2\gamma^2 & \gamma^2 & 0 & 0 \\ . & \gamma^2 & . & . & . \\ . & . & & . & \\ . & & & & \\ 0 & 0 & . \; . \; . & \gamma^2 & 1+2\gamma \end{bmatrix} \quad (5.1)$$

The equation (5.0) can be written

$$\begin{bmatrix} v(1,1) \\ . \\ . \\ . \\ . \\ v(m,1) \end{bmatrix} = A\begin{bmatrix} v(1,0) \\ . \\ . \\ . \\ . \\ v(m,0) \end{bmatrix} + k\begin{bmatrix} f(x_1) \\ . \\ . \\ . \\ . \\ f(x_m) \end{bmatrix} + \frac{1}{2}\gamma^2\begin{bmatrix} v(0.0) \\ . \\ . \\ . \\ . \\ v(m+1,0) \end{bmatrix} \quad (5.2)$$

Stability Analysis for the Wave Equation

Stability for the Finite Difference scheme for the wave equation Consider the following finite difference approximation for the wave equation (Gordon D. Smith, 2004):

By substituting in to the above equation we have

$$\phi k e^{in\Delta x\theta}\phi_{k+1} = \gamma^2\phi k e^{i\Delta x\theta}\phi_k + 2\left(1-\gamma^2\right)\phi k e^{i\Delta x\theta}\phi_k + \gamma^2\phi k e^{-i\Delta x\theta}\phi_k - u_n^{k-1}\phi k e^{in\Delta x\theta}\phi_{k-1} \quad (4.9)$$

$$= \left(\gamma^2 e^{i\Delta x\theta} + 2\left(1-\gamma^2\right) + \gamma^2 e^{-in\Delta x\theta}\right)e^{in\Delta x\theta}\phi_k - ke^{in\Delta x\theta}\phi_{k-1} \quad (5.0)$$

We can cancel the terms and use double angle formulae of trig identity to get

$$\phi_{K+1} = 2\left(1+\gamma^2\left(\cos\Delta x\theta-1\right)\right)\phi_K - \phi_{K-1} \quad (5.1)$$

$$= 2\left(1-2\gamma^2\sin^2\frac{\Delta x\theta}{2}\right)\phi_K - \phi_{K-1} \quad (5.2)$$

For simplicity, we can assume that has the following exponential pattern so the above equation changes to the quadratic equation: $D^2-2\gamma^2 D+1=0$, putting we get (5.3a)

$$D^2-2\gamma^2 D+1=0 \quad (5.3b)$$

Where $\gamma = \left(1-2\gamma^2\sin^2\frac{\Delta x\theta}{2}\right)$. Thus, the solutions of quadratic equation yield

$D_{1,2} = \gamma \pm \sqrt{\gamma^2-1}$ These are two roots. (5.4)

Now having that the roots of above equation is and , conclusively we can write that

$$(D-D_1)(D-D_2) = D^2-(D_1-D_2)D+D_1D_2 = 0 \quad (5.5)$$

Comparing the last terms in these two equations () and () we finalise that

$$D_1D_2 = 1 \quad (5.6)$$

For stability of solution of the form , we need that and . Considering the limit (), the only possibility, if the solutions are stable, is that . Hence must be on the unit disk, which means that

$$|\gamma| \le 1 \qquad (5.7)$$

Thus $\left|1-2\gamma^2 \sin^2 \frac{\Delta x\theta}{2}\right| \le 1$ (5.8)

Or

$$-1 \le 1-2\gamma^2 \sin^2 \frac{\Delta x\theta}{2} \le 1 \qquad (5.9)$$

So that

$$-2 \le -2\gamma^2 \sin^2 \frac{\Delta x\theta}{2} \le 0 \qquad (5.6)$$

The right inequality satisfies automatically, while the leads to the condition

$$\gamma^2 \sin^2 \frac{\Delta x\theta}{2} \le 1 \qquad (5.7)$$

Since maximum value achievable for is 1, we can conclude for stability that

$$\gamma = \left(\frac{c\Delta t}{\Delta x}\right) \le 1 \qquad (5.8)$$

Or

$\Delta t \le \frac{\Delta x}{c}$ (5.8)

Therefore, this has to be satisfied for the stability of finite difference for the wave equation.

Results

Wave Equation in one Dimension

We choose h=0.01and k=0.02. since c=2, this produce =0.1.

Table 4.4: solution of the wave equation (1.26) with its boundary conditions

	x	0	0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2	2.2	2.4	2.6	2.8	3
dt																	
0		0	0.587785	0.951057	0.951057	0.587785	1.23E-16	-0.58779	-0.95106	-0.95106	-0.58779	-1.1E-15	0.587785	0.951057	0.951057	0.587785	-1.4E-15
0.0002		0	0.57656	0.932893	0.932893	0.57656	1.23E-16	-0.57656	-0.93289	-0.93289	-0.57656	-1.1E-15	0.57656	0.932893	0.932893	0.57656	-1.3E-15
0.0004		0	0.543311	0.879096	0.879096	0.543311	1.23E-16	-0.54331	-0.8791	-0.8791	-0.54331	-1.1E-15	0.543311	0.879096	0.879096	0.543311	-1E-15
0.0006		0	0.48931	0.791721	0.791721	0.48931	1.23E-16	-0.48931	-0.79172	-0.79172	-0.48931	-9.3E-16	0.48931	0.791721	0.791721	0.48931	-6.6E-16
0.0008		0	0.416619	0.674104	0.674104	0.416619	1.23E-16	-0.41662	-0.6741	-0.6741	-0.41662	-7.2E-16	0.416619	0.674104	0.674104	0.416619	-2.5E-16
0.001		0	0.328015	0.530739	0.530739	0.328015	1.23E-16	-0.32801	-0.53074	-0.53074	-0.32801	-4.4E-16	0.328015	0.530739	0.530739	0.328015	9E-17
0.0012		0	0.226881	0.367102	0.367102	0.226881	1.17E-16	-0.22688	-0.3671	-0.3671	-0.22688	-1.3E-16	0.226881	0.367102	0.367102	0.226881	3.23E-16
0.0014		0	0.117082	0.189443	0.189443	0.117082	1E-16	-0.11708	-0.18944	-0.18944	-0.11708	1.74E-16	0.117082	0.189443	0.189443	0.117082	4.34E-16
0.0016		0	0.00281	0.004547	0.004547	0.00281	6.98E-17	-0.00281	-0.00455	-0.00455	-0.00281	4.07E-16	0.00281	0.004547	0.004547	0.00281	4.52E-16
0.0018		0	-0.11157	-0.18052	-0.18052	-0.11157	2.48E-17	0.111569	0.180522	0.180522	0.111569	5.26E-16	-0.11157	-0.18052	-0.18052	-0.11157	4.34E-16
0.002		0	-0.22169	-0.3587	-0.3587	-0.22169	-3.4E-17	0.221686	0.358696	0.358696	0.221686	5.14E-16	-0.22169	-0.3587	-0.3587	-0.22169	4.36E-16
0.0022		0	-0.32334	-0.52317	-0.52317	-0.32334	-1E-16	0.323336	0.523169	0.523169	0.323336	3.88E-16	-0.32334	-0.52317	-0.52317	-0.32334	4.91E-16
0.0024		0	-0.41264	-0.66766	-0.66766	-0.41264	-1.6E-16	0.412636	0.667658	0.667658	0.412636	1.89E-16	-0.41264	-0.66766	-0.66766	-0.41264	5.9E-16
0.0026		0	-0.48617	-0.78665	-0.78665	-0.48617	-2.1E-16	0.486174	0.786646	0.786646	0.486174	-2.6E-17	-0.48617	-0.78665	-0.78665	-0.48617	6.89E-16
0.0028		0	-0.54114	-0.87559	-0.87559	-0.54114	-2.4E-16	0.541142	0.875586	0.875586	0.541142	-2E-16	-0.54114	-0.87559	-0.87559	-0.54114	7.17E-16
0.003		0	-0.57544	-0.93108	-0.93108	-0.57544	-2.4E-16	0.57544	0.931081	0.931081	0.57544	-2.8E-16	-0.57544	-0.93108	-0.93108	-0.57544	6.11E-16
0.0032		0	-0.58776	-0.95101	-0.95101	-0.58776	-2E-16	0.587758	0.951013	0.951013	0.587758	-2.7E-16	-0.58776	-0.95101	-0.95101	-0.58776	3.49E-16
0.0034		0	-0.57763	-0.93462	-0.93462	-0.57763	-1.3E-16	0.577626	0.934619	0.934619	0.577626	-2E-16	-0.57763	-0.93462	-0.93462	-0.57763	-4.6E-17
0.0036		0	-0.54543	-0.88253	-0.88253	-0.54543	-4.8E-18	0.545431	0.882526	0.882526	0.545431	-9.2E-17	-0.54543	-0.88253	-0.88253	-0.54543	-5.1E-16
0.0038		0	-0.4924	-0.79672	-0.79672	-0.4924	1.61E-16	0.492402	0.796723	0.796723	0.492402	3.82E-18	-0.4924	-0.79672	-0.79672	-0.4924	-9.3E-16
0.004		0	-0.42057	-0.68049	-0.68049	-0.42057	3.61E-16	0.420565	0.680489	0.680489	0.420565	6.06E-17	-0.42057	-0.68049	-0.68049	-0.42057	-1.2E-15

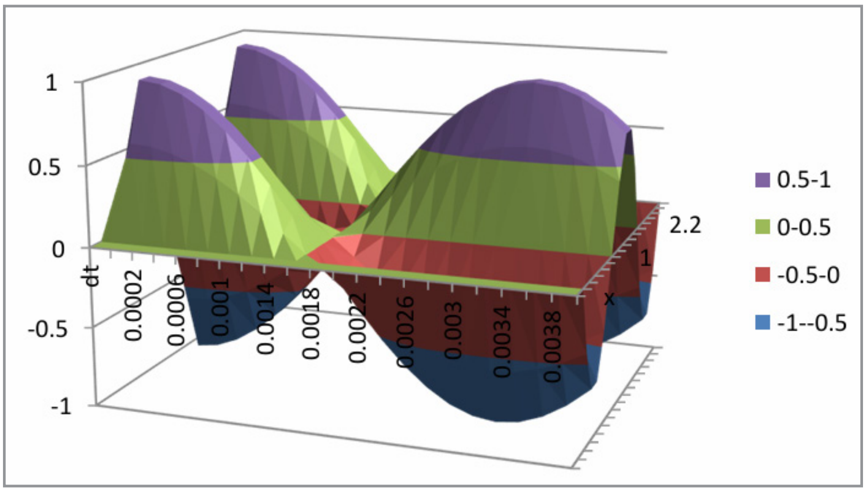

Figure 4.4a: The Elastic String for Equation (1.26)

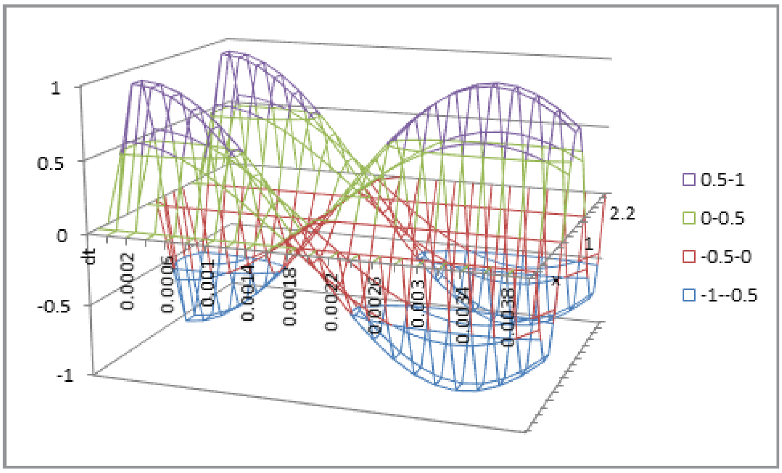

Figure 4.4b

Applying formula (4.6) and (5.0) consecutively to form rows will give approximation to solution given in the table 4.4 for and . A three-dimension representation of the data in table 4.4 is given in figure (4.4a) and (4.4b)

Conclusions

This research has introduced method for solving 1-D linear second order partial differential equation for hyperbolic Equation using finite difference method (FDM). The FDM with three types as the forward, central and backward differences are used to solve heat equation in 1Dimension. Strengths and weaknesses (in term of stability and accuracy) of these methods were checked by numerical solution.

The implicit method is more stable than explicit and the Crank Nicolson is the in term of stability and accuracy. The Crank Nicolson is unconditionally stable. In FDM, the step taking for solving is convergent and accurate. Successive over relaxation method is applied in elliptic equation to speed up the rate of convergence. When this method is applied, the number of iterations reduces drastically. In this research, we consider numerical methods for PDEs and obtain the approximations for finite difference method [6-11].

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Conflict of Interest

The authors declare no conflicts of interest.

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