ISSN: 3067-8307
Review Article

World Journal of Applied Mathematics and Statistics

Euler-Lagrange Equations on Three-Dimensional Almost Kenmotsu Manifolds

Ibrahim Yousif I Abad alrhman* & Gebreel Mohammed khur Baba Gebreel

Department of Mathematics. Faculty of Education. West Kordufan University. Sudan

***Corresponding author:** Ibrahim Yousif I, Department of Mathematics. Faculty of Education. West Kordufan University. Sudan.

Submitted: 10 March 2025 **Accepted:** 17 March 2025 **Published:** 24 March 2025

https://doi.org/10.63620/MKWJAMS.2025.1006

***Citation:** Yousif, I., & Alrhman, A., Baba Gebreel, G. M. K. (2025). Euler-Lagrange Equations on Three-Dimensional Almost Kenmotsu Manifolds. Wor Jour of Appl Math and Sta, 1(1), 01-03.*

Abstract

In this study we concluded the Euler- Lagrange equations on(M^3,ϕ.ξ,η,g), The Three-Dimensional almost Kenmotsu manifolds, have been derived also important applications of Euler-Lagrange mechanical systems. Finally achieved that Three-Dimensional almost Kenmotsu manifolds have this system in Mechanics and Physical Fields as well as in differential geometry.

Keywords: Kenmotsu Manifolds, Three – Dimensional Almost Kenmotsu Manifolds, Euler-Lagrange Equations

Introduction

The geometric study of dynamical systems is an important chapter of contemporary mathematics due to its applications in Mechanics, Theoretical Physics.

There are also a large number of studies on this subject, for example Jun, J and Pathak submitted On Kenmotsu manifolds [1]. and De, U.C. and Pathak obtained on 3-dimensional Kenmotsu manifolds [2].

It is some important work for examples [3-7]. In this paper we will study Kenmotsu manifolds

In this paper, we Euler-Lagrange Equations on Three-Dimensional almost Kenmotsu manifolds. After Introduction in Section 1, we consider Historical Background paper basic. Section 2 deals with the study preliminary. Section 3 is devoted to study 3. three – dimensional Almost Kenmotsu manifolds. Section 4 is devoted to study . Euler-Lagrange Equations on Three-Dimensional almost Kenmotsu manifolds

Preliminary

In this in this preliminary chapter, we recall basic definitions, results and formulas which we shall use in the subsequent chapters of the paper.

Definition (Kenmotsu Manifolds)

Let M^(2n+1) be a (2n+1)-dimensional smooth differentiable manifold (ϕ.ξ,η,g) be an almost contact Riemannian manifold. where is a(1-1) tenser field η is a 1- form and the Riemannian metric .It well known that

$$\phi(\xi) = 0$$

$$\eta(\phi(x)) = 0 \qquad and \qquad \eta(\xi) = 1$$

$$\phi^2(X) = -X + \eta(X)\xi \quad , \quad \phi^2 = -1 + \eta \otimes \xi$$

$$g(X, \xi) = \eta(X)$$

$$g(\phi x, \phi y) = g(x, y) - \eta(x)\eta(y)$$

$$rank\ \phi = n - 1$$

The fundamental 2- form of an almost contact metric manifolds is defined by

ϕ(x,y)=g(x,ϕy)

Lemma 2.2 [2] If ω,θ and k-form be respectively then

- dω ^d θ = -d θ ^ dω
- d=d ∘ d=0
- d(ω^ψ)=dω^ψ+^(-1)^^k dψ^ω

Lemma 2.3 [2] Suppose (U,x_1,…,x_n) is a chart on a manifold. Then

$$\left(\frac{\partial x^j}{\partial x^i}\right) = \delta^i_j = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$$

Three – Dimensional Almost Kenmotsu Manifolds

Definition 3.1 Let. (M^3,ϕ.ξ,η,g) be a 3-dimensional almost Kenmotsu manifold. In what follows, we denote by L=R(.,ξ)ξ ,h=1/2 Lξϕ and h ̅=hϕ , where L denotes the Lie differentiation and R is the Riemannian curvature tensor. From Dileo and Pastore [6, 8]. we see that both h and h0 are symmetric operators and we recall some properties of almost Kenmotsu manifolds as follows:

$$h\xi = I\xi = 0 \quad , \quad tr\ h = tr(\acute{h}) = 0$$

$$h0 = \phi h = 0$$

$$\nabla\xi = h + id - \eta \circ \phi$$

Proposition 3.2 Any 3-dimensional almost Kenmotsu manifold is Kenmotsu if and only if h vanishes.

Definition3.3

Let three-dimensional manifold M=f(x,y,z) R^3,z≠0 ; where (x,y,z) are the standard coordinates in R^3: The vector fields

$$e_1 = \frac{\partial}{\partial x}, \qquad e_2 = \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}$$

are linearly independent at each point of M: Let g be the Riemannian metric defined by

$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0$

$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = 1$

Let η be the 1-form defined by η(Z) = g(Z,e_3) for any Z□ℵ(M).

Proposition 3.4

Let ϕ be the (1,1) tensor field defined by

$\phi(e_1) = -$ e_2

$\phi(e_2) =$ e_1

$\phi(e_3) = 0$

Then using the linearity of ϕ and g we have

η(e_3)= 1 ; ϕ² (z)= -Z +η(Z) e3 ;

g(ϕZ;ϕW)= g(Z;W)-η(Z)η(W);

for any Z,W ϕ(M): Thus for e_3=ξ, (ϕ,ξ,η,g) defines an almost contact metric structure on M: Now, by direct computations we obtain

$[e_1,e_3] = 0 \quad ; [e_2,e_3] = -e_2 \quad ,[e_1,e_3] = -e1$

Proposition 3.5

The vector fields

$$e_1 = \frac{\partial}{\partial x},\ e_2 = \frac{\partial}{\partial y}, \qquad e_3 = \frac{\partial}{\partial z}$$

If ϕ is defined a complex manifoldMthen ϕ2=ϕϕ=-1

Proof.

$$\phi^2(e_1) = \phi(\phi(e_1)) = \phi(-e_2) = -\phi(e_2) = -e_1$$

$$\phi^2(e_2) = \phi(\phi(e_2)) = \phi(e_1) = \phi(e_1) = -e_2$$

As can ϕ2 is -1 (complex) or 0

Euler-Lagrange Equations on Three-Dimensional Almost Kenmotsu Manifolds

In this section, we shall obtain the version Euler-Lagrange equations for classical mechanics structured with Three-Dimensional almost Kenmotsu manifolds introduced in Let semispray be a vector field as follows

$$\xi = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \quad , \quad x = \dot{x} \ , \quad y = \dot{y} \ , \quad z = \dot{z}$$

By Liouville vector field on Three-Dimensional almost Kenmotsu manifolds space form (M^3,ϕ.ξ,η,g), we call the vector field determined by V=ϕξ and calculated by

$$\phi\xi = \phi\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) = x\phi\left(\frac{\partial}{\partial x}\right) + y\phi\left(\frac{\partial}{\partial x}\right) + z\phi\left(\frac{\partial}{\partial x}\right)$$

$$\phi\xi = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} + z(0) = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} + z(0)$$

$$\phi\xi = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$$

Denote T by the kinetic energy and P by the potential energy of mechanics system on Three-Dimensional almost Kenmotsu manifolds. Then we write by L=T-P Lagrangian function and by E_L = V (L)- L

the energy function associated L. Operator i_J defined by

iϕ:2 M31 M3

is called the interior product with ϕ, or sometimes the insertion operator, or contraction by ϕ.The exterior vertical derivation d_ϕ is defined by

d_ϕ=[i_ϕ,d]=i_ϕ d-di_ϕ

where d is the usual exterior derivation? For almost product structure ϕ determined by the closed Three-Dimensional almost Kenmotsu manifolds form is the closed 2-form given by ϕ_L = -dd_ϕ L such that

$$d_\phi = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}$$

$$d_\phi L = \left(-x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x}\right)L = -x\frac{\partial L}{\partial y} + y\frac{\partial L}{\partial x}$$

Thus we get

$$\phi_L = -d(d_\phi L) = -d\left(-x\frac{\partial L}{\partial y} + y\frac{\partial L}{\partial x}\right)$$

$$\phi_L = X\frac{\partial^2 L}{\partial x\,\partial y}dx\wedge dx + X\frac{\partial^2 L}{\partial y\,\partial y}dy\wedge dy - Y\frac{\partial^2 L}{\partial x\,\partial x}dx\wedge dy - Y\frac{\partial^2 L}{\partial y\,\partial y}dy\wedge dy - Z\frac{\partial^2 L}{\partial x\,\partial z}dx\wedge dz$$
$$- -Z\frac{\partial^2 L}{\partial y\,\partial z}dy\wedge dz$$

Because of the closed Three-Dimensional almost Kenmotsu manifolds form ϕ_L on Three-Dimensional almost Kenmotsu manifolds space form (M^3,ϕ.ξ,η,g) is para-symplectic structure, one may obtain

$$E_L = X\frac{\partial L}{\partial Y} + Y\frac{\partial L}{\partial x} - L$$

Considering (0,1) we calculate

$$dE_L = d\left(X\frac{\partial L}{\partial Y} + Y\frac{\partial L}{\partial x} - L\right)$$

$$dE_L = X\frac{\partial^2 L}{\partial x\partial y}dx + y\frac{\partial^2 L}{\partial y\partial y}dx + X\frac{\partial^2 L}{\partial x\partial x}dy + Y\frac{\partial^2 L}{\partial y\partial x}dy - \frac{\partial L}{\partial x}dx - Z\frac{\partial^2 L}{\partial x\partial z}dx - Z\frac{\partial^2 L}{\partial y\partial z}dy - \frac{\partial L}{\partial y}dy$$

Taking care of i_ξ ϕ_L = dE_L, we have

$$X\frac{\partial^2 L}{\partial x\partial y}dx + y\frac{\partial^2 L}{\partial y\partial y}dx + Z\frac{\partial^2 L}{\partial y\partial z}dx - \frac{\partial L}{\partial x}dx + X\frac{\partial^2 L}{\partial x\partial x}dy + Y\frac{\partial^2 L}{\partial y\partial x}dy + Z\frac{\partial^2 L}{\partial x\partial z}dy + \frac{\partial L}{\partial y}dy = 0$$

$$\frac{\partial L}{\partial y}\left(X\frac{\partial}{\partial x}dx + y\frac{\partial}{\partial y}dx + Z\frac{\partial}{\partial z}dx\right) - \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial x}\left(X\frac{\partial}{\partial x}dy + Y\frac{\partial}{\partial y}dy + Z\frac{\partial}{\partial z}dy\right) + \frac{\partial L}{\partial y}dy = 0$$

$$\frac{\partial L}{\partial y}\left(X\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)dx - \frac{\partial L}{\partial x}dx + \frac{\partial L}{\partial x}\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right)dy + \frac{\partial L}{\partial y}dy = 0$$

$$\left[\frac{\partial L}{\partial y}\left(X\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) - \frac{\partial L}{\partial x}\right]dx + \left[\frac{\partial L}{\partial x}\left(X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}\right) + \frac{\partial L}{\partial y}\right]dy = 0$$

If the curve α I R → M^3 be integral curve of ξ,

$$\alpha = \frac{\partial}{\partial t} = X\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z}$$

which satisfies

$$\left[\frac{\partial}{\partial t}\frac{\partial L}{\partial y} - \frac{\partial L}{\partial x}\right]dx + \left[\frac{\partial}{\partial t}\frac{\partial L}{\partial x} + \frac{\partial L}{\partial y}\right]dy = 0$$

it follows equations

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial y} - \frac{\partial L}{\partial x} = 0\ , \quad \frac{\partial}{\partial t}\frac{\partial L}{\partial x} + \frac{\partial L}{\partial y} = 0$$

so-called Euler-Lagrange equations whose solutions are the paths of the semispray ξ on Three-Dimensional almost Kenmotsu manifolds space form (M^3,ϕ.ξ,η,g). Finally, one may say that the triple (M^3,ϕ.ξ,η,g) is mechanical system on Three-Dimensional almost Kenmotsu manifolds (M^3,ϕ.ξ,η,g) Therefore we say

References

1. Jun, J.-B., De, U. C., & Pathak, G. (2005). On Kenmotsu manifolds. Journal of the Korean Mathematical Society, 42, 435-445.
2. De, U. C., & Pathak, G. (2004). On 3-dimensional Kenmotsu manifolds. Indian Journal of Pure and Applied Mathematics, 35, 159-165.
3. Kirichenko, V. F. (2001). On the geometry of Kenmotsu manifolds. Doklady Akademii Nauk, 380, 585-587.
4. Wang, Y. (2016). A class of 3-dimensional almost Kenmotsu manifolds with harmonic curvature tensors. Open Mathematics, 14, 977-985.
5. Saltarelli, V. (2015). Three-dimensional almost Kenmotsu manifolds satisfying certain nullity conditions. Bulletin of the Malaysian Mathematical Sciences Society, 38, 437-459.
6. Chaubey, S. K., & Yadav, S. K. (2018). Study of Kenmotsu manifolds with semi-symmetric metric connection. Universal Journal of Mathematics and Applications, 1(2), 89-97.
7. De, U. C., & Pathak, G. (2004). On 3-dimensional Kenmotsu manifolds. Indian Journal of Pure and Applied Mathematics, 35, 159-165.