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A Comparative Study of Gauss-Seidel Method and Pseudo Inversion Method in Option Pricing

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Abstract

The paper seeks to compare the use of Gauss-Seidel method and Pseudo Inversion method for evaluating American option under Black-Scholes model, through a drifted financial derivative system, discretized from Black-Scholes financial PDE. In particular, we conducted a numerical analysis of the methods in order to give a better understanding of the numerical problems associated with the valuation of the options. Some numerical difficulties are discussed by illustrative example.

Keywords: Financial PDE; Stochastic Algorithm; Drifted Financial Derivative System; Option Pricing; Central Finite Difference Discretization; Gauss-Seidel Method; Pseudo Inversion Symmetric Positive Semidefinite. (MSC) Subject Classification: 60H15, 65kxx, 65Axx, 65Cxx

Introduction

In this paper we investigate the application of Gauss-Seidel method and Pseudo Inversion method to Black-Scholes option pricing models. In particular, we concentrate the attention on the difficulties that arise in the discretized financial matrix that is expected to be symmetric positive (semi) definite and stable which fails to be so. A well-known example occurs in an indefinite Hessian in optimization by Gil, Murray, and Wright and financial matrix in Solomon for pricing an American option [1-2].

Customarily, the early exercise nature of American option makes its pricing numerical. One of the idea is to formulate a linear complementarity problem (LCP) for the price and then solve it numerically after discretization. Another way is to discretize the Black-Scholes differential equation into system of ordinary differential equation and further transform into a drifted financial derivative system and then solve numerically using stochastic approximation method (SAM), Pseudo-Inverse Method (PIM) and Gauss-Seidel method. In the failure detection and failure identification areas for the past years, numerous approaches have been developed to control law reconfiguration. One of them, the Pseudo-Inverse Method (PIM), has been accepted as a key approach to reconfigurable control and it has been used quite successfully in flight simulations as reported by [3- 6]. Modification of the feedback gain so that the reconfigured system approximates the nominal system in some sense was the key idea, but some methods require more than that for its accuracy. In this study, Gauss-Seidel method and Pseudo inversion method are investigated on an American Option under the Black-Scholes model to establish their relationship and differences in implementation.

The work was arranged as follows; In section 2.1, Black-Scholes model was presented, the partial differential equation which financial derivative have to satisfy and we discretize the generic PDE into LCP and drift financial derivative system for American option valuation. Then in section 2.2, we describe two finite difference schemes (Gauss-Seidel method and Pseudo inverse method) and their properties. Numerical experiments are presented in section 3.1 and conclusions are given in section 3.2.

The Model

We assume that the dynamics of the underlying asset is described by a standard Geometric Brownian Motion (GBM) diffusion process for the underlying seen in equation (1). We now assume a market consisting of a single risky asset (S) and a risky-free bank account (r). This market is given by the equations:

$$dS = \mu S dt + \sigma S dZ, \tag{1}$$

called the geometric Brownian-Motion and

$$dB = rBdt, \tag{2}$$

called the non-stochastic, where Z is Brownian motion, B is the bond value and the interpretation of the parameters is as follows:
μ is the expected rate of return in the risk asset (drift),
$\sigma > 0$, is the volatility of the risky asset,
$r \geq 0$, is the bank's rate of interest.
The value of the parameters μ and σ may be estimated from historical data, having, μ as the mean return of S, and σ the variance. The quantity dZ is a normally distributed random variable with mean 0 and variance dt.

$$dZ \propto N(0, (\sqrt{dt})^2),$$

having that each interval dt,dZ is a sample drawn from the distribution $N(0, (\sqrt{dt})^2)$, and multiplied by σ to produce the term σdZ.

According to, Black and Scholes and Merton it is shown that the worth V(t,S) of any contingent claim written on a stock, whether it is American or European, satisfies the famous Black-Scholes equation[7-9].

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0, \tag{3}$$

where volatility σ, the risk-free rate r, and dividend yield q are all assumed to be constants. The terminal and boundary conditions determines the value of any particular contingent claims. It can easily be noted that the PDE only holds in the not-yet-exercised region in American option. At the place where the option should be exercised immediately, the equality sign in (3) would turn into an inequality one. That means the option value $V(t,S)$ at each time follows either $V(t,S) = \Lambda(t,S)$ for the early exercised region or (3) for the not-yet-exercised region, where $\Lambda(t,S)$ is the payoff of an American option at time t.

Changing the variable $\tau = T - t$, we the generic form of (3) as,

$$\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)\, S\frac{\partial V}{\partial S} + rV = 0, \tag{4}$$

where $V(\tau,.) \equiv V(T-\tau,.)$, $\sigma(\tau,.) \equiv \sigma(T-\tau,.), 0 \le \tau \le T$ $S_{min} < S < S_{max}$, subject to the initial condition:

$$V(0,S) = \Lambda(S). \tag{5}$$

The unbounded domain is truncated to

$$(t,S) \in (S,0) \times (0,T] \tag{6}$$

with sufficiently large $S \equiv S_{max}$ for computations.

Under Black-Scholes model it is easy to verify that the worth V of an American option satisfies an LCP

$$\begin{cases} LV \ge 0 \\ V \ge \Lambda \\ (LV)(V-\Lambda) = 0, \end{cases} \tag{7}$$

with the boundary conditions

$$\begin{cases} V(t,0) = 0 \\ \qquad\qquad S \in (0, S_{max}) \\ V(t,S) = \Lambda(t,S), \end{cases} \tag{8}$$

Beyond the boundary $S = S_{max}$, $V(t,S) = \Lambda(S)$ for $S \ge S_{max}$, that is the worth V is approximated to be the same as the payoff Λ .

Pde Discretization In Finance

The free boundary nature of American options make its exercise open at any time before expiry. Formally, the value of an American put option with a strike price k is

$$V(0,k) = \sup(0 \le \tau^* \le T : E(e^{-\tau\tau^*}(k - S_{\tau*})^+), \tag{9}$$

were τ* is the optimal exercise time that maximizes the expected payoff - any scheme to price an American options must calculate this (τ*).

The equivalent of equation (4) for American options with payoff Λ(S) is

$$\begin{bmatrix} \frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV \ge 0 \\ V(T,S) \ge \Lambda(S) \end{bmatrix} \tag{10}$$

$$\left[\frac{\partial V}{\partial \tau} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - (r-q)S\frac{\partial V}{\partial S} + rV\right][V(T,S) - \Lambda(S)] = 0.$$

Using a uniform spatial mesh on the interval $[S_{min}, S_{max}]$:

$$S_j = j\delta S + S_{min}, \ j = 0,1,\dots,n+1, \tag{11}$$

where

$$\delta S = \frac{S_{max} - S_{min}}{n+1}, \text{ and} \tag{12}$$

$$S_{max} = S_0 \exp\left[6\sigma\sqrt{T} + \left(r - q - \frac{\sigma^2}{2}\right)T\right]. \tag{13}$$

It can be validated that the truncated domain (6) has the lower bound $S_{min} = 0$ and upper bound S_{max} as in (13).

Hence, replacing all derivatives with respect to S by their central finite-difference approximations, we obtain the following approximation to the Black-Scholes PDE (4)

$$\frac{\partial V(\tau,S)}{\partial \tau} = \frac{1}{2}\sigma^2 S^2 \frac{V(\tau,S_j+\delta S) - 2V(\tau,S_j) + V(\tau,S_j-\delta S)}{\delta S^2} + (r-q)S\frac{V(\tau,S_j+\delta S) - V(\tau,S_j-\delta S)}{2\delta S} - rV(\tau,S) + O(\delta S^2). \tag{14}$$

Let $V_j(\tau)$ denote the semi-discrete approximation to $V(\tau,S_j)$. We obtain the following system of first-order ordinary differential equations, by applying (14) at each internal node S_j,

$$\frac{dV_j(\tau)}{d\tau} = \underbrace{\frac{1}{2}\left(\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 - \frac{(r-q)S_j}{\delta S}\right)}_{L_{j,j-1}} V_{j-1}(\tau) + \underbrace{\left(-\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 - r\right)}_{L_{j,j}} V_j(\tau) + \underbrace{\frac{1}{2}\left(\left(\frac{\sigma(S_j)S_j}{\delta S}\right)^2 + \frac{(r-q)S_j}{\delta S}\right)}_{L_{j,j+1}} V_{j+1}(\tau)$$

$$j = 1,2,\dots,n; \tag{15}$$

Furthermore, it can reduced to a discretized form given as

$$\frac{dV_j(\tau)}{d\tau} = L_{j,j-1}V_{j-1}(\tau) + L_{j,j}V_j(\tau) + L_{j,j+1}V_{j+1}(\tau).$$

System (15) has n equations in n+2 unknown functions, $V_0(\tau), V_1(\tau), \dots, V_n(\tau), V_{n+1}(\tau)$. Using Dirichlet boundary conditions we have the functions $V_0(\tau)$ and $V_{n+1}(\tau)$ which respectively approximate the solution at the boundary nodes $S_0 = S_{min}$ and $S_{n+1} = S_{max}$. For computation, system (15) can be re-written into matrix-vector differential equation with an n-by-n tri-diagonal coefficient matrix L whose entries are defined in (15) to,

$$\frac{d\boldsymbol{V}(\tau)}{d\tau} = L\boldsymbol{V}(\tau) + \boldsymbol{G}(\tau), \tag{16}$$

subject to the initial condition (5)

$$\boldsymbol{V}(0) = \boldsymbol{\Lambda} := [\Lambda(S_1), \Lambda(S_2), \dots, \Lambda(S_n)]'. \tag{17}$$

$$L = \begin{pmatrix} L_{11} & L_{12} & 0 \cdots & 0 & 0 \\ L_{21} & L_{22} & L_{23} \cdots & 0 & 0 \\ 0 & L_{32} & L_{33} \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots & \vdots \\ 0 & 0 & 0 \cdots & L_{n-1,n-1} & L_{n-1,n} \\ 0 & 0 & 0 \cdots & L_{n,n-1} & L_{n,n} \end{pmatrix}, \ V(\tau) = \begin{pmatrix} V_1(\tau) \\ V_2(\tau) \\ \vdots \\ V_{n-1}(\tau) \\ V_n(\tau) \end{pmatrix}.$$

The vector $G(\tau) \in R^n (G(\tau)$ contains boundary values of the mesh solution) is given by

$$\left[\left(\frac{\sigma^2(S_0)\,S_0^2}{2\delta S^2} - \frac{(r-q)S_0}{2\delta S}\right) V_0(\tau), 0, \dots, 0, \left(\frac{\sigma^2(S_{n+1})S_{n+1}{}^2}{2\delta S^2} + \frac{(r-q)S_{n+1}}{2\delta S}\right) V_{n+1}(\tau)\right]',$$

From (16), according to Duffy [8], it was reaffirmed that PDE techniques allow us to create a framework for modeling complex and interesting derivatives products.
The spatial discretization leads to:

Semi-discrete Linear Complementarity Problem

From the work of White [10] using (10), (16) and (17), we have

$$\begin{cases} L^j\, V^{j+1} \geq g^j \\ V^{j+1} \geq \Lambda \\ (V^{j+1} - \Lambda)'(L^j\, V^{j+1} - g^j) = 0 \end{cases}, \qquad (18)$$

having L as n-by-n tri-diagonal coefficient matrix, g a vector resulting from the second term in equation (16) V and Λ vectors containing the grid point values of the worth V and the pay off Λ, respectively. The solution must be obtained at every time step. For crude approximation it is just to solve the system L^j V= g^j, then set L^{j+1} = max (V, Λ).

Drifted financial derivative system

By using the work of Shibli, it can be seen that G(τ) term in (16) can be treated as an enforced input to the financial derivative system, resulted from boundary condition defined in (8). With zero boundary condition, equation (16) yields [11].

$$\dot{V} = Lv \qquad (19)$$

which represents a pfaffian differential constraints see "Luca and Oriolo, [12]", but not of kinematic nature, arises from the conservation on non-zero financial derivatives [12]. The transformed financial derivative system (19) can be re-expressed as

$$Lv = d\,. \qquad (20)$$

where, $L \in R^{n\times n}$ is a coefficient matrix, $d \in R^n$ is a known column of constants and v is the unknown vector. System (20) represents a drifted financial derivative system with a drift term d. In such a system the derivative value V has been solved by Osu and Solomon using stochastic algorithm, Solomon using Pseudo-Inverse Method and Solomon using Gauss-Seidel method, but here we propose a comparative study of Gauss-Seidel method and Pseudo Inverse method [13,2,14].

Controllability And Stability Of Financial Derivatives

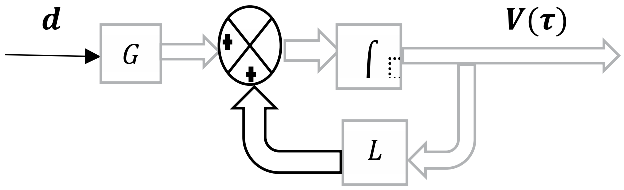

Figure 1: Open-loop financial controlled system

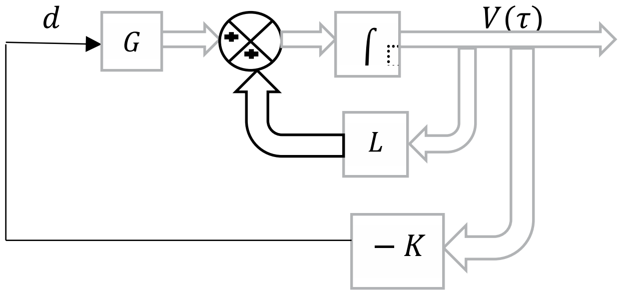

Figure 2: Closed-loop controlled financial system with $w = -Kv$

Consider the following differential state equation

$$\frac{dV(\tau)}{d\tau} = LV(\tau) + wG(\tau), \qquad (21)$$

where,
V – is a state vector
L – is n x n matrix
G – is n x 1
w – is a control signal
τ – time .

In theory of linear time-invariant dynamical control systems the most popular and the most frequently used mathematical model is given by (21). The continuous-time system (21 and 16) shown in figure 1, is said to be state controllable at τ=τ_0 if it is possible to construct an unconstrained control signal that will transfer an initial state to any final state in a finite time interval $\tau_0 \leq \tau \leq \tau_1$. The system is said to be controllable if and only if the following n x n matrix is full rank n,

$$\hat{M} = [G \;\; LG \;\; L^2G \dots L^{n-1}G] \qquad (22)$$

The matrix is called the controllability matrix.

Let

$$w = -Kv. \qquad (23)$$

be the control signal which is determined by an instantaneous state,

$$\hat{T} = \hat{M}\, W \qquad (24)$$

the transformation matrix where $\hat{M}$ is the controllability matrix (22) and

$$W = \begin{pmatrix} a_{n-1} & a_{n-2} & \cdots & a & 1 \\ a_{n-2} & a_{n-3} & \vdots & 1 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_1 & 1 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \end{pmatrix}, \qquad (25)$$

where the a_i's are the coefficients of the characteristic polynomial

$$|hI - L| = h^n + a_1 h^{n-1} + \cdots + a_{n-1}h + a_n. \quad (26)$$

Let us choose a set desired eigenvalues as $h_1 = u_1, h_2 = u_2, \ldots, h_n = u_n$. Then the desired characteristic equation becomes

$$(h - u_1)(h - u_2).\cdots.(h - u_n) = h^n + \alpha_1 h^{n-1} + \cdots + \alpha_{n-1}h + \alpha_n. \quad (27)$$

The sufficient condition for the system to be completely controllable with all eigenvalues arbitrarily placed is by choosing the gain matrix

$$K = [(\alpha_n - a_n)(\alpha_{n-1} - a_{n-1}) \cdots (\alpha_2 - a_2)(\alpha_1 - a_1)]\hat{T}^{-1}. \quad (28)$$

Finite Difference Technics

According to the PDE technics as described in section 2, there is need to decide which numerical scheme to adopt. For this reason, we first recall some results from numerical analysis of finite difference method and examine in details how the numerical schemes usually adopted in finance can cope with the discontinuity of the initial condition as in Gianluca Fusai et al [15].

The Gauss-Seidel Method

The exact solution of the system (20) is denoted by v=L^(-1) d, when the matrix of the coefficients is non-singular. It is known that direct methods for solving such systems requires about n^3/3 operations which is not suitable for large sparse systems. Iterative methods appear to be the appropriate choice especially when the convergence of the method up to the required accuracy is achieved within n steps. See Meligy and Youssef [16].

In Gauss-Seidel method, we use the new values v_i^((k+1)) as soon as they are known. For example, once we have computed $v_1^{(k+1)}$ from the first equation, its value is then used in the second equation to obtain the new $v_2^{(k+1)}$ and so on. Here we applied Gauss-Seidel method on a refined financial matrix to pricing American option as follows:

Let L be a matrix as in (16), the Gauss-Seidel method of L is defined as a matrix $L^s \in M$ (M is a vector space of m×n matrices) with drift term d, satisfying all the following criteria:

1. L^s has no zeros on its main diagonal, if any of the diagonal entries are zero, then rows or columns must be interchanged to obtain a coefficient matrix that has nonzero entries on the main diagonal.

2. L^s must be a strictly diagonally dominant matrix, for which convergence is guaranteed.

The matrix, L^sexists for any matrix, L. Hence, we demonstrate the method as follows; for each k≥1, we generate the component $v_i^{(k)}$ of v^k from $v^{(k-1)}$ by

$$v_i^{(k)} = \frac{1}{l_{ii}^s}\left[-\sum_{j=1}^{i-1}\left(l_{ij}^s v_j^{(k)}\right) - \sum_{j=i+1}^{n}\left(l_{ij}^s v_j^{(k-1)}\right) + d_i\right], \quad (29)$$

$$for\ i = 1,2,\ldots,n$$

Namely,

$$\begin{aligned} l_{11}^s v_1^{(k)} &= l_{12}^s v_2^{(k-1)} - \cdots - l_{1n}^s v_n^{(k-1)} + d_1 \\ l_{11}^s v_1^{(k)} + l_{22}^s v_2^{(k)} &= -l_{23}^s v_3^{(k-1)} - \cdots - l_{2n}^s v_n^{(k-1)} + d_2 \\ &\vdots \\ l_{n1}^s v_1^{(k)} + l_{n2}^s v_2^{(k)} + \cdots l_{nn}^s v_n^{(k-1)} &= d_n \end{aligned} \quad (30)$$

Matrix form of Gauss-Seidel method.

The above equation (30) can be re-express as

$$L^s v = d, \quad (31)$$

and transformed into

$$(D-L-U)x = d, \quad (32)$$

where D= diagonal matrix, L= lower triangular matrix and U= upper triangular matrix. Hence, from (30) using (32), we

$$v^{(k)} = (D - L)^{-1}Uv^{(k-1)} + (D - L)^{-1}d \quad (33)$$

Defined $A_g = (D - L)^{-1}U$ and $b_g = (D - L)^{-1}d$, Gauss-Seidel method can be written as

$$v^{(k)} = A_g v^{(k-1)} + b_g, \quad k = 1,2,3,\ldots \quad (34)$$

Numerical Algorithm of Gauss-Seidel Method

Input: $L^s = [l^s_{ij}]$, $d, V_0 = v^{(0)}$, tolerance TOL, maximum number of iterations N.

Step 1. Set $k = 1$

Step 2. While ($k \leq N$) do step 3-6

Step 3. for $i = 1,2,\ldots,n$

$$v_i^{(k)} = \frac{1}{l_{ii}^s}\left[-\sum_{j=1}^{i-1}\left(l_{ij}^s v_j^{(k)}\right) - \sum_{j=i+1}^{n}\left(l_{ij}^s v_j^{(k-1)}\right) + d_i\right]$$

Step 4. If $\|v - V0\| < TOL, then\ OUTPUT\ (v_1, v_2, v_3, \ldots, v_n)$;

STOP.

Step 5. Set $k = k + 1$,

Step 6. $for\ i = 1,2,\ldots,n$

set $V0i = v_i$.

Step 7. OUTPUT $(v_1, v_2, v_3, \ldots, v_n)$;

STOP.

Convergence theorem of the iteration methods

Let the iteration method be written as

$$v^{(k)} = Av^{(k-1)} + b, \quad for\ each\ k = 1,2,3,\ldots$$

Lemma 3.1

If the spectral radius satisfies $\rho(A) < 1$, then $(I-A)^{-1}$ exist, and $(I-A)^{-1} = I + A + A^2 + \cdots = \sum_{j=0}^{\infty} A^j$

Theorem 3.1

For any $v^{(0)} \in R^n$, the sequence $\{v^{(k)}\}_{k=0}^{\infty}$ defined by $v^{(k)} = Av^{(k-1)} + b$, $for\ each\ k \geq 1$ converges to the unique solution of v=Av+b if and only if $\rho(A) < 1$.

Proof

Let $v^{(k)} = Av^{(k-1)} + b = A(Av^{(k-1)}+b)+b = \cdots = A^k v^{(0)} + (A^{(k-1)}+\cdots+A+I)$,

$$\text{since } \rho(A) < 1, \lim_{k\to\infty} A^k v^{(0)} = 0,$$

$$\lim_{k\to\infty} v^k = 0 + \lim_{k\to\infty}\left(\sum_{j=0}^{k-1} A^j\right)b = (I-A)^{-1}\, b.$$

Theorem 3.2

If L^s is strictly diagonally dominant, then for any choice of $v^{(0)}$, Gauss-Seidel method gives a sequence $\{v^{(k)}\}_{k=0}^{\infty}$ that converges to the unique solution of $L^s\, v = d$.

The Frobenius Norm Positive Approximant

The following theorem gives the solution to the problem of positive approximation in the Frobenius norm.

Theorem 3.3

Let $A \in R^{n\times n}$, and let $B=(A+A^T)/2$ and $C=(A-A^T)/2$ be the symmetric and skew-symmetric parts of A respectively. Let B=UH be a polar decomposition ($U^T U = I$, $H = H^T \geq 0$). Then $X_F=(B+H)/2$ is the unique positive approximant of A in the Frobenius norm, and $\delta_F(A)^2 = \sum_{\lambda_i(B)<0} \lambda_i(B)^2 + \|C\|_F^2$, as seen in Higham [17].

Norm Positive Approximant

Halmos , proves the following result, in the more general context of linear operators on a Hilbert space [18].

Theorem 3.4

For $A \in R^{n\times n}$,

$$\delta_2(A) = \min\{r \geq 0 : r^2 I + C^2 \geq 0 \text{ and } B + (r^2 I + C^2)^{\frac{1}{2}} \geq 0\}, \quad (35)$$

where $B=(A+A^T)/2$ and $C=(A-A^T)/2$ be the symmetric and skew-symmetric parts of A respectively. The matrix

$$P = B + (\delta_2(A)^2 I + C^2)^{\frac{1}{2}} \quad (36)$$

is a positive approximant of A.

The importance of Halmos' result is that it replaces the problem of minimizing over the set of n×n symmetric positive semidefinite matrices by the much simpler problem of minimizing over the nonnegative scalars.

It should be stressed that a 2-norm positive approximant of A is not in general unique. To see this, note that

$$B = (A+A^T)/2 \quad (37)$$

Is a nearest symmetric matrix to A in the 2-norm, Fan and Hoffman [19]. If B is symmetric positive semidefinite, then clearly it must also be a positive approximant for A; but if C^2 is not a multiple of the identity, then B differs from the positive approximant given by (36).

The Pseudo-Inverse Method

Over the years, pseudo-inverse of a matrix has been used by many researchers in the Reconfigurable Control system (RCS) with a considerable success. Here we applied pseudo inverse of a matrix to pricing American option as follows:

Let L be a matrix as in (16 and 29), the pseudo-inverse of L is defined as a matrix $L^+ \in M$ (M is a vector space of m×n matrices) satisfying all the following criteria:

3. $LL^+ L = L$ (LL^+ need not be the general identity matrix, but it maps all column vectors of L to themselves);
4. $L^+ LL^+ = L^+$ (L^+ is a weak inverse for the multiplicative semi-group);
5. $(LL^+)' = LL^+$ (LL^+ is Hermitian and $(LL^+)'$ is the transpose of Hermitian); and
6. $(L^+ L)' = L^+ L$ ($L^+ L$ is also Hermitian).

L^+exists for any matrix, L.

But when the latter has full rank (n), L^+can be expressed as a simple algebraic formula.

When L has linearly independent columns (and thus matrix $L'L$ is invertible), L^+ can be computed as:

$$L^+ = (L'L)^{-1} L', \quad (38)$$

this particular pseudo inverse constitutes a left inverse, since, in this case, $L^+ L = I$.

When L has linearly independent rows (matrix LL' is invertible), L^+ can be computed as:

$$L^+ = L'(L'L)^{-1}. \quad (39)$$

This is a right inverse, as

$LL^+ = I$.

According to Shibli [11], the derivative value V can be gotten from the drifted finacial derivative system (20) by computing the pseudo inversion matrix (39) expressed in the form,

$$V = L^+ d. \quad (40)$$

However, the solution from (40) does not necessarily make the closed- loop controlled financial system in figure 2, stable which can reduce the effectiveness of the method. But alternatively, the system can be stabilized if the controllability condition (22) is satisfied, therefore making implementation of pole placement design a key ingredient to stabilize the system.

Numerical Experiment

In this section, we illustrate the two methods mentioned early to price American put options, using parameters in White derive from Black-Scholes model and they are defined below [10]:

Table 1: Estimated parameters for the Black-Scholes model

Parameter	Notation	Value
Risk free interest rate	r	0.2
Dividend yield	q	0.1
Strike price	K	7
Volatility	σ	0.3
Time to expiry	T	2
Spot price	S_0	10
Ratio of nodes	ϑ	30

We illustrate the method in a concrete setting, by first plugging the parameters in table 1 in (11, 12 and 13), with the time nodes $3x10^3$ and space nodes $9x10^4$ satisfying the ratio of nodes ϑ as stipulated, we have S_{max}=142.33, and the space discretization steps as

$$\delta S = \frac{142.33}{90001} = 0.002.$$

Thus, from (11, 12 and 15) we have;

$$S_1 = 0.002,\ L_{11} = 0.2,\ L_{12} = 0.05,$$
$$S_2 = 0.004,\ L_{21} = -0.1,\ L_{22} = 0.2,\ L_{23} = 0.1,$$
$$S_3 = 0.006,\ L_{32} = -0.15,\ L_{33} = 0.2,$$

and then the financial matrix (3 by 3 tri-diagonal coefficient matrix).

$$L = \begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix}. \quad (41)$$

By using the equation of total investment return;

$$r = d+q, \quad (42)$$

where

r: is the risk adjusted discount rate for V (the worth)

q: is the dividend yield (or convenience yield in case of commodities) and

d: is the drift (or capital gain rate).

Hence d=0.1 for $q = 0.1$ and $d = 0.2$ for $q = 0.0$ (No dividend yield).

Application Of Pseudo Inversion Mtheod

From (20), we have

$$\begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}. \quad (43)$$

Using the financial matrix (41) in (39), we have

$$L^+ = \begin{pmatrix} -0.4 & 0.8 & 0.6 \\ 0.06 & -0.4 & 1.1 \\ 1.5 & 0.9 & 0.5 \end{pmatrix}.$$

From (40), we have

$$V = L^+ d = \begin{pmatrix} -0.4 & 0.8 & 0.6 \\ 0.06 & -0.4 & 1.1 \\ 1.5 & 0.9 & 0.5 \end{pmatrix} \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}$$
$$= (0.2 \quad 0.152 \quad 0.58).$$

Applying the inversion matrix L^+ to (40) gives, V_1 = (0.2 0.152 0.58) and V* (S,t) = 0.31 for both values of the drift. The above result is not equal to the solution obtained in White [10]. It is desired to check the controllability condition (22).

For n=3 in (22), we have

$[G\ \ LG\ \ L^2G] = \hat{M}$,

where G and L are defined as in (21) and (41) respectively.

$$L^2 = \begin{pmatrix} 0.035 & 0.02 & 0.005 \\ -0.04 & 0.02 & 0.04 \\ 0.015 & -0.06 & 0.025 \end{pmatrix}, \quad G = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

with little explanation it can be seen that the controllability matrix

$$\hat{M} = \begin{pmatrix} 1 & 0.25 & 0.06 \\ 1 & 0.2 & 0.02 \\ 1 & 0.05 & -0.02 \end{pmatrix}, \quad (44)$$

is of full rank 3. The controllability condition been satisfied, implies that the pole placement design can be implemented to stabilize the system.

From (26)

$$|hI - L| = \left| \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix} - \begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix} \right|$$

$$|hI - L| = h^3 - 0.6h^2 + 0.14h - 0.01. \quad (45)$$

With n=3 in (26), we have

$$|hI - L| = h^3 + a_1 h^2 + a_2 h + a_3. \quad (46)$$

Comparing (45) and (46), we have; a_1=-0.6, a_2=0.14, a_3=-0.01.

Hence, from (25) we have

$$W = \begin{pmatrix} 0.14 & -0.6 & 1 \\ -0.6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (47)$$

Substituting (44) and (47) into the transformation matrix (24), we have

$$\hat{T} = \begin{pmatrix} 1 & 0.25 & 0.06 \\ 1 & 0.2 & 0.02 \\ 1 & 0.05 & -0.02 \end{pmatrix} \begin{pmatrix} 0.14 & -0.6 & 1 \\ -0.6 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

$$\hat{T} = \begin{pmatrix} 0.05 & -0.35 & 1 \\ 0.04 & -0.4 & 1 \\ 0.09 & -0.55 & 1 \end{pmatrix}.$$

$$|\hat{T}| = 0.004$$

$$\hat{T}^{-1} = \frac{1}{0.004} \begin{pmatrix} 0.15 & -0.2 & 0.05 \\ 0.05 & -0.04 & -0.01 \\ 0.014 & -0.004 & -0.006 \end{pmatrix}$$

$$\hat{T}^{-1} = \begin{pmatrix} 37.5 & -50 & 12.5 \\ 12.5 & -10 & -2.5 \\ 3.5 & -1 & -1.5 \end{pmatrix}. \quad (48)$$

Placing the pole $h_1 = -1$, $h_2 = -2$, $h_3 = -3$, the desired characteristic equations from (27) is illustrated as

$$|hI - A| = \left| \begin{pmatrix} h & 0 & 0 \\ 0 & h & 0 \\ 0 & 0 & h \end{pmatrix} - \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix} \right|,$$

where A is the pole placement diagonal matrix.

$$|hI-A| = h^3 + 6h^2 + 11h + 6 \quad (49)$$

From (49), we have $\alpha_1 = 6$, $\alpha_2 = 11$, $\alpha_3 = 6$.
For the system to be completely controllable with all eigenvalues arbitrarily placed we choose the gain matrix (28).

Substituting a_i's and α_i's according to (45) and (49) and (48) into (28), we have

$$K = [(6 + 0.01) \quad (11 - 0.14) \quad (6 + 0.6)] \begin{pmatrix} 37.5 & -50 & 12.5 \\ 12.5 & -10 & -2.5 \\ 3.5 & -1 & -1.5 \end{pmatrix}$$

$$K = [384.23 \quad -415.7 \quad 38.07]. \quad (50)$$

Normalizing the gain matrix yields

$$K = [3.8423 \quad -4.157 \quad 0.3807]. \quad (51)$$

According to Shibli [11] as shown in Figure 2, the negative feedback controlled financial system implies that to stabilize such a system, the risk free rate r should be decreased by 3.8423 times (from 0.2 to −0.77), the volatility should also be decrease by 0.3808 times (from 0.3 to -0.11). The drift parameter d should increase the stock by 4.157 times (from 0.2 to 0.8), and from financial point of view, the negative sign is to balance the increase of the stock and comply with the conservation of financial money.

Using the result from the above analysis on stability condition to the pseudo inversion method (40) by the pseudo inverse L^+ we have

$$V = L^+ d$$

$$= \begin{pmatrix} -0.4 & 0.8 & 0.6 \\ 0.06 & -0.4 & 1.1 \\ 1.5 & 0.9 & 0.5 \end{pmatrix} \begin{pmatrix} 0.8 \\ 0.8 \\ 0.8 \end{pmatrix}$$

$$= (0.8 \quad 0.608 \quad 2.32).$$

V_1 = (0.8 0.608 2.32) and V* (S,t) = 1.2427 (V* (S,t) ≈ 1.2). This procedure explains and established that pseudo inversion method can be applied on a discretized financial PDE to price an American option and European option with a considerable success.

Application Of Gauss-Seidel Method

From (20), we have

$$\begin{pmatrix} 0.2 & 0.05 & 0 \\ -0.1 & 0.2 & 0.1 \\ 0 & -0.15 & 0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}. \quad (52)$$

Using numerical algorithm of Gauss-Seidel method defined above in (52), we have the table 2.

Table 2: Iterations for (52) using Gauss-Seidel method

Iteration	v_1	v_2	v_3
1	1	1.5	2.125
2	0.625	0.25	1.187
3	0.938	0.875	1.656
4	0.781	0.562	1.422
5	0.859	0.719	1.539
6	0.82	0.641	1.48
7	0.84	0.68	1.51
8	0.83	0.66	1.495
9	0.835	0.67	1.502
10	0.833	0.665	1.499
11	0.834	0.667	1.501
12	0.833	0.666	1.5
13	0.833	0.667	1.5

From table 2, we have the solution as V_{13} = (0.833 0.667 1.5) for both values of the drift, and is not a fixed point and V* (S,t) =1, also is not equal to the solution in [8, 13 and 16]. It is desired to check the financial matrix in the system (52) to see if is strictly diagonally dominant as required to apply the Gauss-Seidel method.
The financial matrix in the system (52) is not strictly diagonally dominant in the second row
{i.e. |0.2|>|-0.1|+|0.1| is not true} as required to apply the Gauss-Seidel method. Hence, from theorem 3.4, and by (37) we have that the symmetric part of the financial matrix L, can be express as

$$L^s = \frac{(L + L^T)}{2}$$

$$L^s = \begin{pmatrix} 0.2 & -0.025 & 0 \\ -0.025 & 0.2 & -0.025 \\ 0 & -0.025 & 0.2 \end{pmatrix} \quad (53)$$

From (31), we have the refined system

$$\begin{pmatrix} 0.2 & -0.025 & 0 \\ -0.025 & 0.2 & -0.025 \\ 0 & -0.025 & 0.2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0.2 \\ 0.2 \\ 0.2 \end{pmatrix}. \qquad (54)$$

Using numerical algorithm of Gauss-Seidel method defined above in the refined system (54), we have the table 3.

Table 3: Iterations for (54) using Gauss-Seidel method

Iteration	v_1	v_2	v_3
1	1	1.125	1.141
2	1.141	1.285	1.161
3	1.161	1.29	1.161
4	1.161	1.29	1.161

From table 3, we have the solution as V_4=(1.161 1.29 1.161) for both values of the drift, and is not a fixed point and V^* (S,t) = 1.204, (V* (S,t) ≈ 1.2) is now equal to the solution in [8, 13 and 16]. This shows that Gauss-Seidel method can be applied to price an American option and European option with a considerable success through a discretized financial PDE.

Conclusions

In this paper we have illustrated the problems encountered when we apply two numerical scheme (Pseudo Inversion method and Gauss-Seidel method) to the solution of PDE's arising in finance. In particular, we evaluate the financial matrix derive from the Black-Scholes option pricing model, using Pseudo Inversion method and Gauss-Seidel method. The results of our study can be summarized in few points:

a. The pseudo inversion method proves to be simple in pricing American options, but needs the system to be stabilized for its accuracy. To guarantee stability, the controllability condition must be satisfied, before the pole placement design can be implemented to stabilize the system.
b. The Gauss-Seidel method proves to be simple in pricing American options, but equally needs the system to be refined into strictly diagonally dominant system. To guarantee effective result, the financial matrix needs to be refined to a strictly diagonally dominant matrix using 2-norm and Frobenius norm theorem from Nicholas J. Higham.
c. It was ascertaining that both methods can be successfully used to price an American option, but required different conditions on the derived financial matrix from Black-Scholes PDE, for its accuracy.

References

1. Gill P E Murray, Wright M H (1981) Practical Optimization. Academic London.
2. Solomon O U, (2017) A Simple Pseudo Inversion Method for Solution to PDE with Financial Application. Scholars Bulletin 3: 283-291.
3. Caglayan A K, Allen, S M, Wehmuller K (1988) Evaluation of a second generation reconfiguration strategy for aircraft flight control systems subjected to actuator failure/surface damage. In Aerospace and Electronics Conference, 1988. NAECON 1988., Proceedings of the IEEE 1988 National (pp. 520-529). IEEE.
4. Huber RR, McCulloch B (1984) Self-repairing flight control system (No. 841552). SAE Technical Paper.
5. Ostroff, AJ (1985) Techniques for accommodating control effector failures on a mildly statically unstable airplane. In American Control Conference, 1985 (pp. 906-913). IEEE.
6. Rattan KS (1985) Evaluation of control mixer concept for reconfiguration of flight control system. In IEEE. National Aerospace Electronics Conference Proceedings.
7. Black F, Scholes M (1973) The pricing of options and corporate liabilities. Journal of political economy, 81(3), 637-654.
8. Duffy DJ (2013) Finite Difference Methods in Financial Engineering: A Partial Differential Equation Approach. John Wiley & Sons.
9. Merton RC (1973) Theory of rational option pricing. The Bell Journal of economics and management science 141-183.
10. White R (2013) Numerical solution to PDEs with financial applications. Open Gamma Quantitative Research, 10.
11. Shibli M (2012) Dynamics and Controllability of Financial derivatives: Towards Stabilization of the Global Financial System Crisis. Journal of Mathematical Finance, 2(01): 54-65.
12. Luca D, Oriolo G (1995) Modeling and Control of Non-holomic Mechanical Systems. In: J.A. Kecskemethy, Ed., Kinematics and Dynamics of Multi-Body Systems, CISM. Courses and lectures, No. 360, Springer-Verlage 277-342.
13. Osu, B O, Solomon OU (2015) A Simple Stochastic Algorithm for Solution to PDE with Financial Application. Journal of the Nigerian Association of Mathematical Physics 32: 125-132.
14. Solomon O U (2022) Application of Gauss-Seidel Method on Refined Financial Matrix for Solution to Partial Differential Equation (PDE) in Finance. Asian Journal of Pure and Applied Mathematics 4: 531-542.
15. Gianluca, F Simona S, Aldo T (2002) Practical Problems in the Numerical Solution of PDE's in Finance. Rendiconti per gli studi Economici Quantitative Universita Ca' Foscari 105-132
16. Meligy S A and Youssef I K, (2022). Relaxation parameters and composite refinement techniques. Results in Applied Mathematics, (No. 100282) 15: 1-9.
17. Higham NJ (1988) Computiong a Nearest Symmetric Positive Semidefinite Matrix. Elsevier Science Publishing co. inc. New York 103-118.
18. Halmos PR (1972) Positive Approximants of Operators. Indian Unv. Maths Journal. 21: 951-960.
19. Fan, I L and A J Hoffman AJ, (1955). Some Metric Inequalities in the Space of Matrices. Proc. Amer. Math. Soc 6: 111-116.